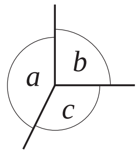
15 Polygons

15.1 Angle Facts

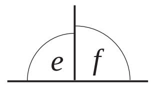
In this section we revise some basic work with *angles*, and begin by using the three rules listed below:

The angles at a point add up to $360°$, e.g.



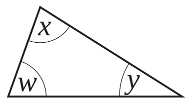
$$a + b + c = 360°$$

The angles on a straight line add up to $180°$, e.g.



$$e + f = 180°$$

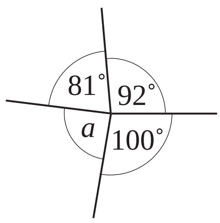
The angles in a triangle add up to $180°$, e.g.



$$w + x + y = 180°$$

Example 1

Determine the size of angle a in the diagram shown.

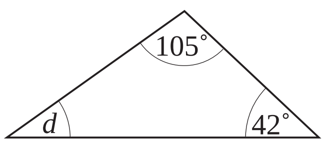


Solution

$$\begin{aligned} 81° + 92° + 100° + a &= 360° \quad \textit{(angle sum at a point)} \\ a + 273° &= 360° \\ a &= 87° \end{aligned}$$

Example 2

Determine the size of angle d in the diagram shown.

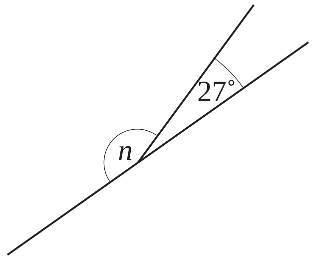


Solution

$$\begin{aligned} 105° + 42° + d &= 180° \quad \textit{(angle sum in a triangle)} \\ 147° + d &= 180° \\ d &= 33° \end{aligned}$$

Example 3

Determine the size of angle n in the diagram shown.



Solution

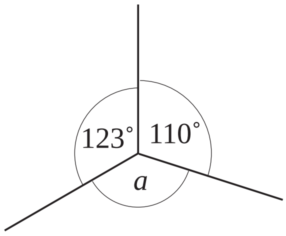
$$n + 27° = 180° \quad \textit{(angle sum on a line)}$$

$$n = 153°$$

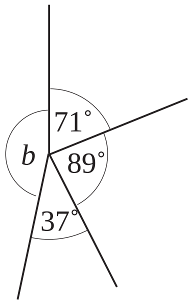
Exercises

1. Calculate the sizes of the angles marked by letters in the following diagrams:

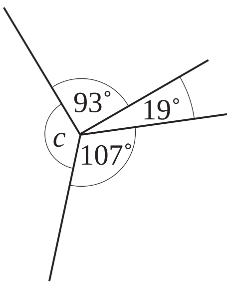
(a)



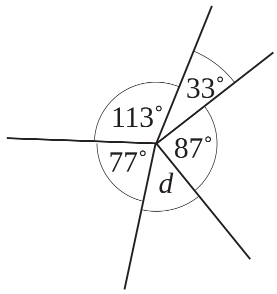
(b)



(c)

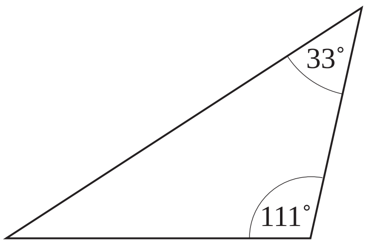


(d)

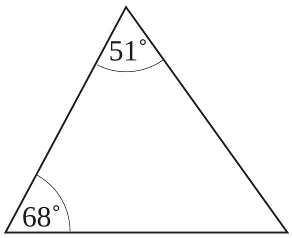


2. Calculate the sizes of the unknown angles in the following triangles:

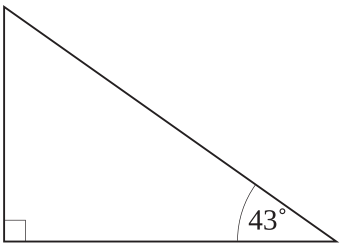
(a)



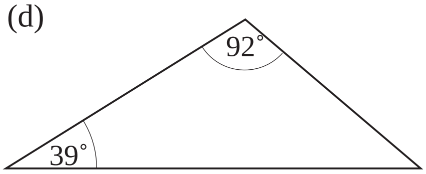
(b)



(c)

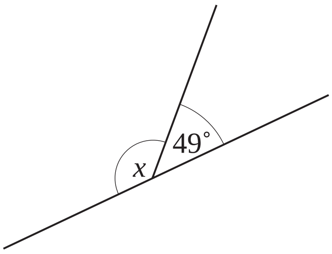


(d)

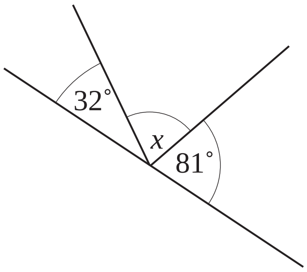


3. Calculate the sizes of the angles marked by the letter x in the following diagrams:

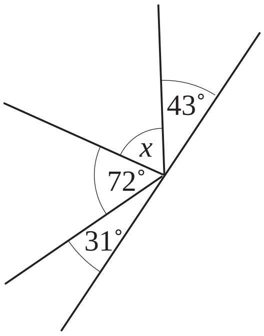
(a)



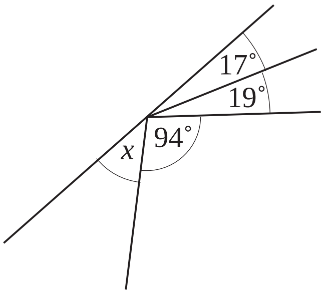
(b)



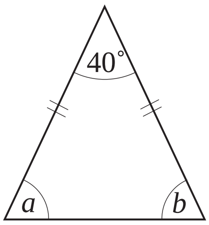
(c)



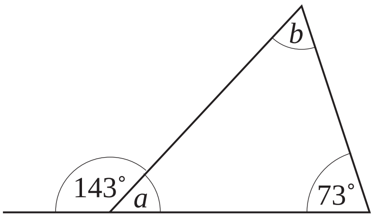
(d)



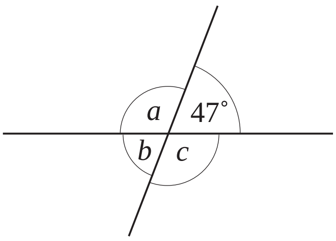
4. The diagram shows an isosceles triangle.
What are the sizes of the two angles marked a and b ?



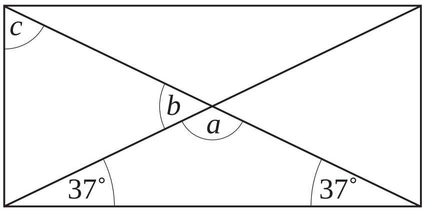
5. Calculate the sizes of the angles marked a and b in the diagram.



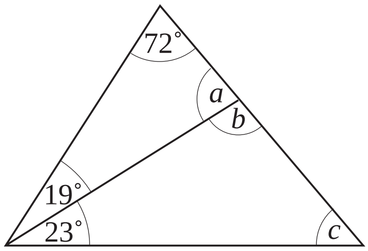
6. The diagram opposite shows two intersecting straight lines. Calculate the sizes of the angles marked a, b and c in the diagram.
What do you notice about angles a and c ?



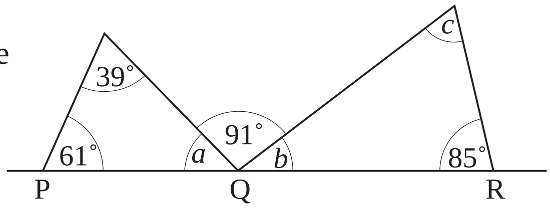
7. The diagram opposite shows a rectangle and its diagonals. Calculate the sizes of the angles marked a, b and c.



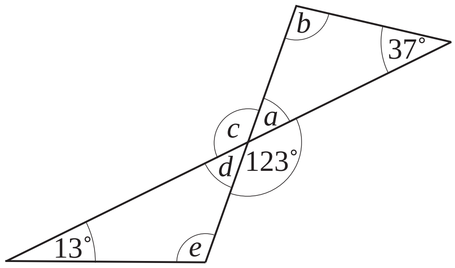
8. Determine the sizes of the angles marked *a*, *b* and *c* in the diagram shown.



9. PQR is a straight line. Determine the sizes of the angles marked *a*, *b* and *c* in the triangles shown.



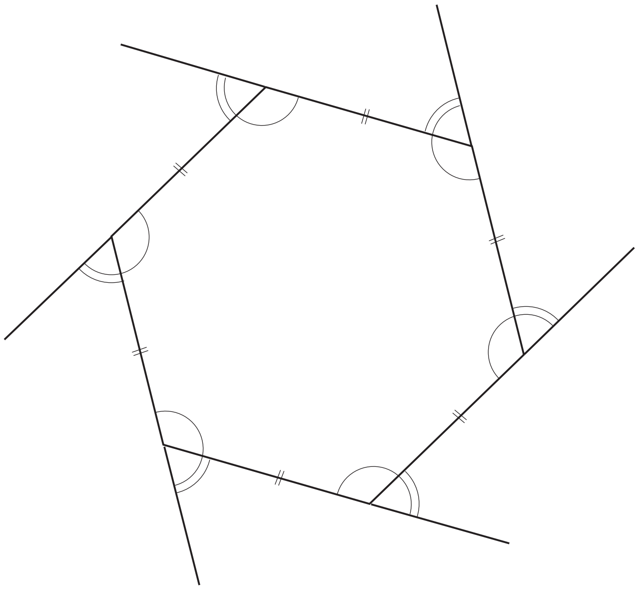
10. Calculate the sizes of the angles marked *a*, *b*, *c*, *d* and *e* in the triangles shown.



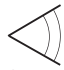
15.2 Angle Properties of Polygons

In this section we calculate the size of the *interior* and *exterior* angles for different regular polygons.

The following diagram shows a regular hexagon:



The angles marked  are the *interior* angles of the hexagon.

The angles marked  are the *exterior* angles of the hexagon.

In a regular polygon the sides are all the same length and the *interior* angles are all the same size.

> Note that, for any polygon:
>
> interior angle + exterior angle = $180°$.

Since the interior angles of a regular polygon are all the same size, it follows that the exterior angles are also equal to one another.

One complete turn of the hexagon above will rotate any one exterior angle to each of the others in turn, which illustrates the following result:

> The exterior angles of *any* polygon add up to $360°$.

Example 1

Calculate the sizes of the *interior* and the *exterior* angles of a regular hexagon. Hence determine the sum of the interior angles.

Solution

The exterior angles of a regular hexagon are all equal, as shown in the previous diagram.

Therefore the exterior angle of a regular hexagon $= \dfrac{360°}{6}$

$= 60°$

So the interior angle of a regular hexagon $= 180° - 60°$

$= 120°$

The sum of the interior angles $= 6 \times 120°$

$= 720°$

Example 2

The *exterior* angle of a regular polygon is $40°$.

Calculate:

(a) the size of the *interior* angle,

(b) the number of sides of the polygon.

Solution

(a) Interior angle $+$ exterior angle $= 180°$

Interior angle $= 180° - 40°$

$= 140°$

(b) The number of sides can be determined by dividing $360°$ by the size of the exterior angles, giving

$$\frac{360}{40} = 9$$

so the polygon has 9 sides.

In a regular polygon:

$$\text{exterior angle} = \frac{360°}{\text{the number of sides}}$$

$$\text{number of sides} = \frac{360°}{\text{exterior angle}}$$

Exercises

1. Calculate the size of the *exterior* angles of a regular polygon which has interior angles of:

   (a) 150 °

   (b) 175 °

   (c) 162 °

   (d) 174 °

2. Calculate the sizes of the *exterior* and *interior* angles of:

   (a) a regular octagon,

   (b) a regular decagon.

3. (a) Calculate the size of the *interior* angles of a regular 12-sided polygon.

   (b) What is the sum of the *interior* angles of a regular 12-sided polygon?

4. (a) What is the size of the *interior* angle of a regular 20-sided polygon?

   (b) What is the sum of the *interior* angles of a regular 20-sided polygon?

5. Calculate the size of the *exterior* angle of a regular pentagon.

6. The size of the exterior angle of a regular polygon is 12 °. How many sides does this polygon have?

7. Calculate the number of sides of a regular polygon with interior angles of:

   (a) (i) 150 ° (ii) 175 °

   (iii) 162 ° (iv) 174 °

   (b) Show why it is impossible for a regular polygon to have an interior angle of 123 °.

8. (a) Complete the following table for regular polygons. Note that many of the missing values can be found in the examples and earlier exercises for this unit.

Number of Sides	*Exterior Angles*	*Interior Angles*	*Sum of Interior Angles*
4	$90°$		
5			
6			
7			
8			
9			
10			
12			

(b) Describe an alternative way to calculate the sum of the interior angles of a regular polygon.

(c) Draw and measure the angles in some irregular polygons. Which of the results in the table are also true for irregular polygons?

9. The exterior angle of a regular polygon is $4°$.

(a) How many sides does the polygon have?

(b) What is the sum of the interior angles of the polygon?

10. A regular polygon has n sides.

(a) Explain why the exterior angles of the polygon are of size $\frac{360°}{n}$.

(b) Explain why the interior angles of the polygon are $180° - \frac{360°}{n}$.

(c) Write an expression for the sum of the interior angles.

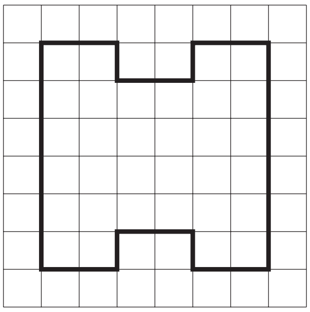
15.3 Symmetry

In this section we revise the *symmetry* of objects and examine the symmetry of regular polygons.

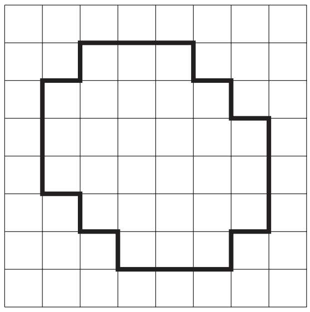
Example 1

Draw the lines of symmetry of each shape below:

(a)

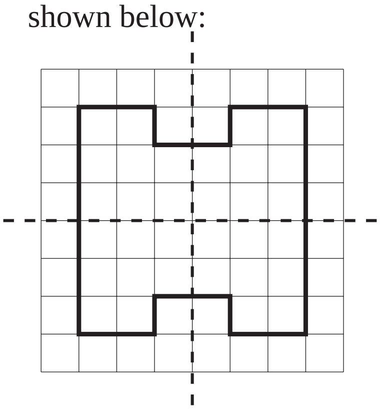


(b)

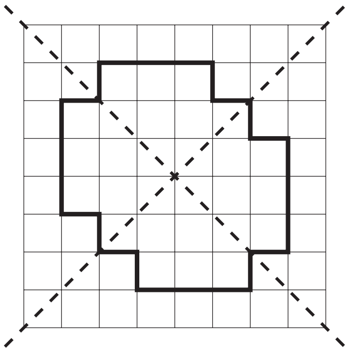


Solution

(a) The shape has 2 lines of symmetry, one horizontal and the other vertical, as shown below:



(b) The shape has 2 diagonal lines of symmetry, as shown below:

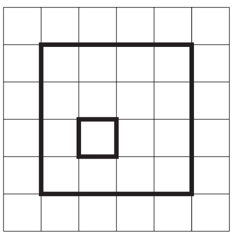


> *Reminder*
>
> The order of rotational symmetry is the number of times in one rotation of $360°$ that a shape is identical to that of its starting position.

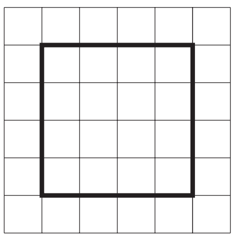
Example 2

What is the order of rotational symmetry of each of the following shapes:

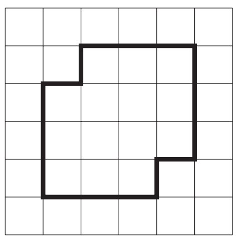
(a)



(b)



(c)

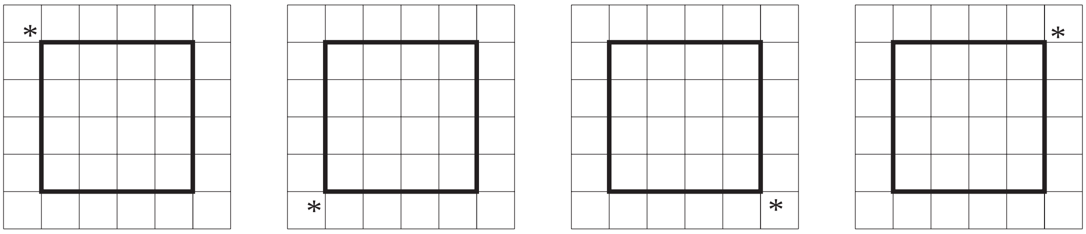


Solution

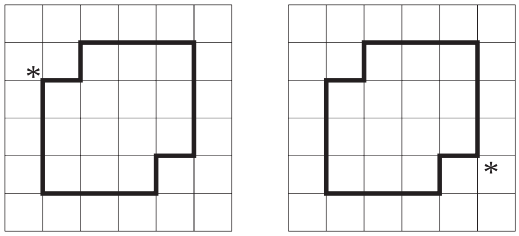
(a) The shape has rotational symmetry of order 1, meaning that it does not have rotational symmetry. (The shape cannot be rotated to another position within $360°$ and still look the same.)

(b) The shape has rotational symmetry of order 4.

The following diagram shows how the position of one corner, marked *, moves as the square is rotated anticlockwise about its centre.



(c) The shape has rotational symmetry of order 2. The diagram shows the position of a corner, marked *, as the shape is rotated about its centre.



Example 3

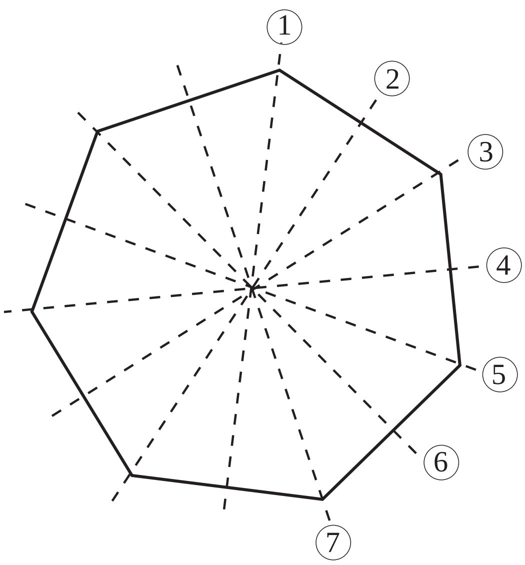
A heptagon is a shape which has 7 sides.

(a) Draw a diagram to show the lines of symmetry of a regular heptagon.

(b) What is the order of rotational symmetry of a regular heptagon?

Solution

(a) A regular heptagon has 7 lines of symmetry, as shown in the following diagram:



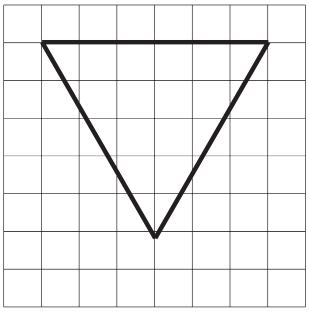
(b) A regular heptagon has rotational symmetry of order 7.

> The order of rotational symmetry and the number of lines of symmetry of *any regular* polygon is equal to the number of sides.

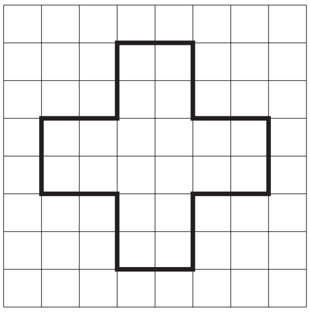
Exercises

1. Copy each of the following shapes and draw in all the lines of symmetry. For each one, state the order of rotational symmetry and mark on your copy its centre of rotation.

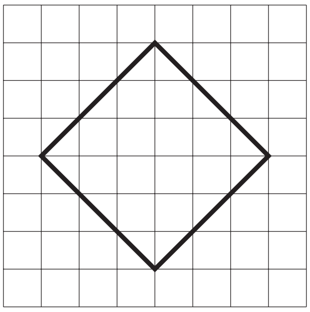
(a)



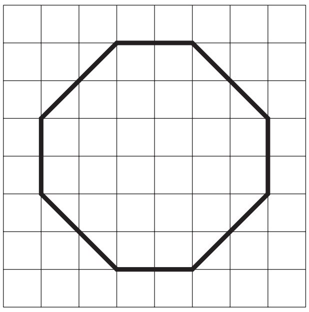
(b)



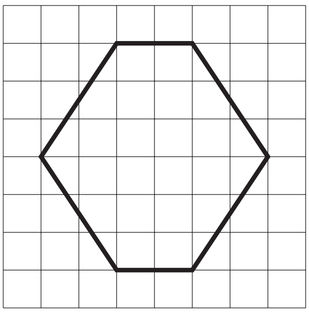
(c)



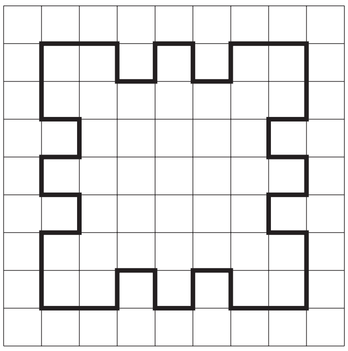
(d)



(e)

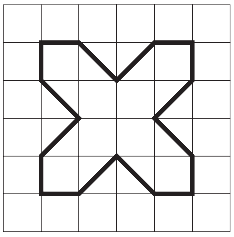


(f)

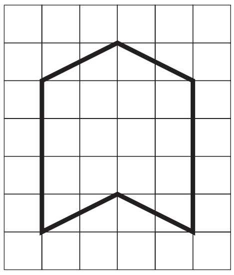


2. State the order of rotational symmetry and the number of lines of symmetry, for each of the following shapes:

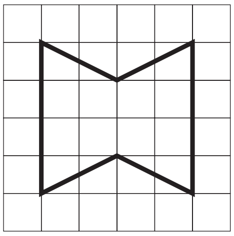
(a)



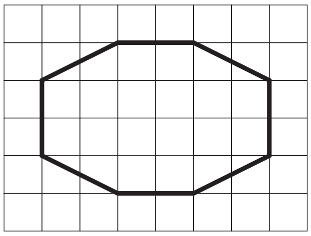
(b)



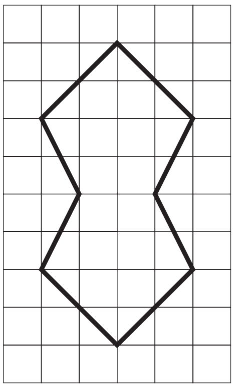
(c)



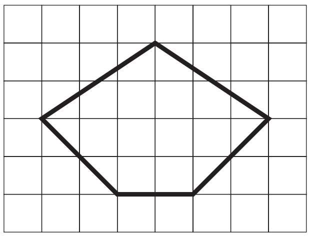
(d)



(e)

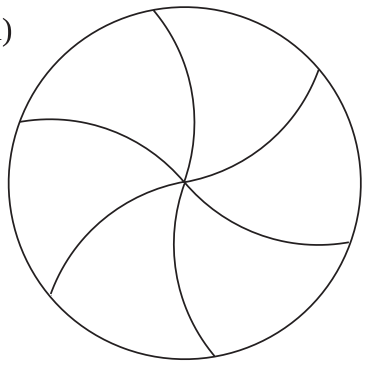


(f)

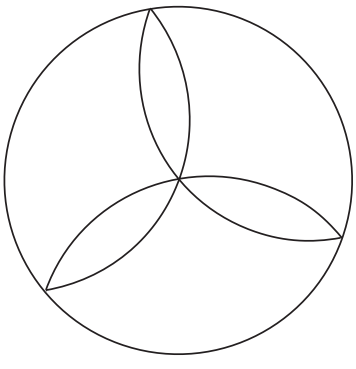


3. Describe fully the symmetries of the following shapes:

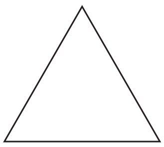
(a)



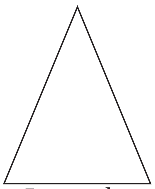
(b)



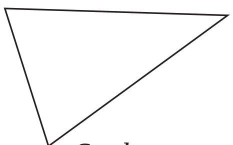
4. Describe the symmetry properties of each of the following triangles:



Equilateral



Isosceles



Scalene

5. (a) How many lines of symmetry does a *square* have?
Draw a diagram to show this information.

(b) What is the order of rotational symmetry of a *square*?

6. (a) Copy and complete the following table:

Shape	*Order of Rotational Symmetry*	*Number of Lines of Symmetry*
Equilateral triangle		
Square		
Regular pentagon		
Regular hexagon		
Regular heptagon (7 sides)		
Regular octagon		
Regular nonagon (9 sides)		
Regular decagon (10 sides)		
Regular dodecagon (12 sides)		

(b) What do you conclude from the information in the table?

7. Draw a shape that has *no* lines of symmetry, but has rotational symmetry of order 3.

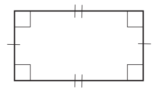
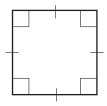
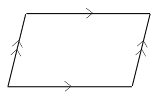
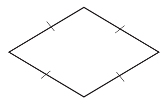
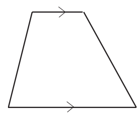
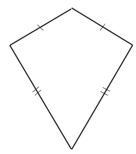
8. Draw a shape that has *at least one* line of symmetry and *no* rotational symmetry.

9. Draw two regular polygons, one with an *even* number of sides and one with an *odd* number of sides. By drawing lines of symmetry on each diagram, show how the positions of the lines of symmetry differ between *odd*- and *even-sided* regular polygons.

10. Draw an irregular polygon that has both *line* and *rotational* symmetry. Show the lines of symmetry and the centre of rotation, and state its order of rotational symmetry.

15.4 Quadrilaterals

There are many special types of quadrilaterals; the following table lists some of them and their properties.

Quadrilateral	*Properties*	
Rectangle	4 right angles and opposite sides equal	
Square	4 right angles and 4 equal sides	
Parallelogram	Two pairs of parallel sides and opposite sides equal	
Rhombus	Parallelogram with 4 equal sides	
Trapezium	Two sides are parallel	
Kite	Two pairs of adjacent sides of the same length	

Example 1

List the quadrilaterals that have four sides all of the *same length*.

Solution

Square and *rhombus*.

Example 2

List the quadrilaterals that do *not* have two pairs of parallel sides.

Solution

Kite and *trapezium*.

Example 3

Which quadrilaterals have diagonals that are *perpendicular* to one another?

Solution

The *square*, *rhombus* and *kite* have diagonals that cross at right angles.

Exercises

1. Which quadrilaterals have diagonals that are the same length?

2. (a) Which quadrilaterals have *exactly two* lines of symmetry?

   (b) Draw diagrams to show these lines of symmetry.

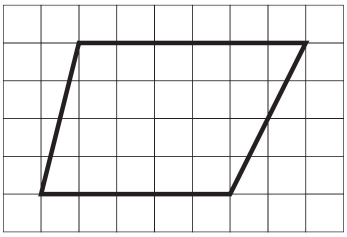
3. Which quadrilaterals have rotational symmetry of order 2 ?

4. (a) Which quadrilaterals can have *exactly one* line of symmetry?

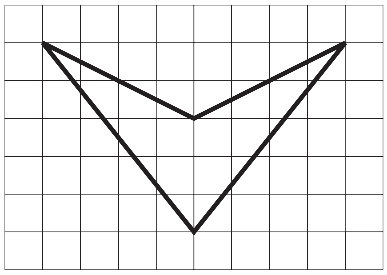
   (b) Draw diagrams to show them and the line of symmetry.

5. Name each of the following quadrilaterals:

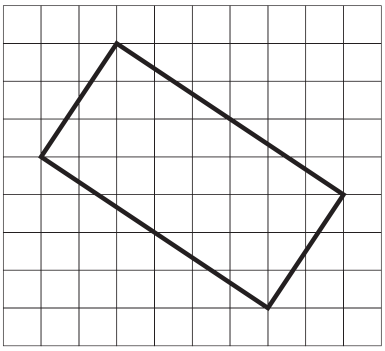
(a)



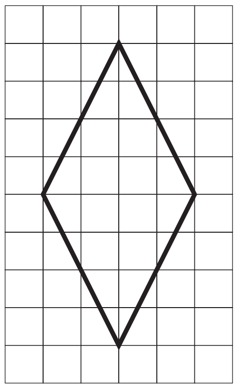
(b)



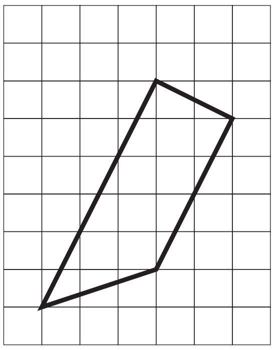
(c)



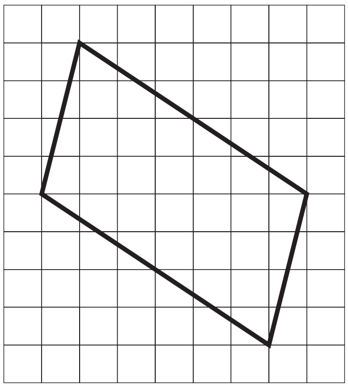
(d)



(e)



(f)



6. Which quadrilaterals have diagonals that are *not* equal in length?

7. A quadrilateral has four sides of the same length. Copy and complete the following sentences:

   (a) The quadrilateral must be a ....

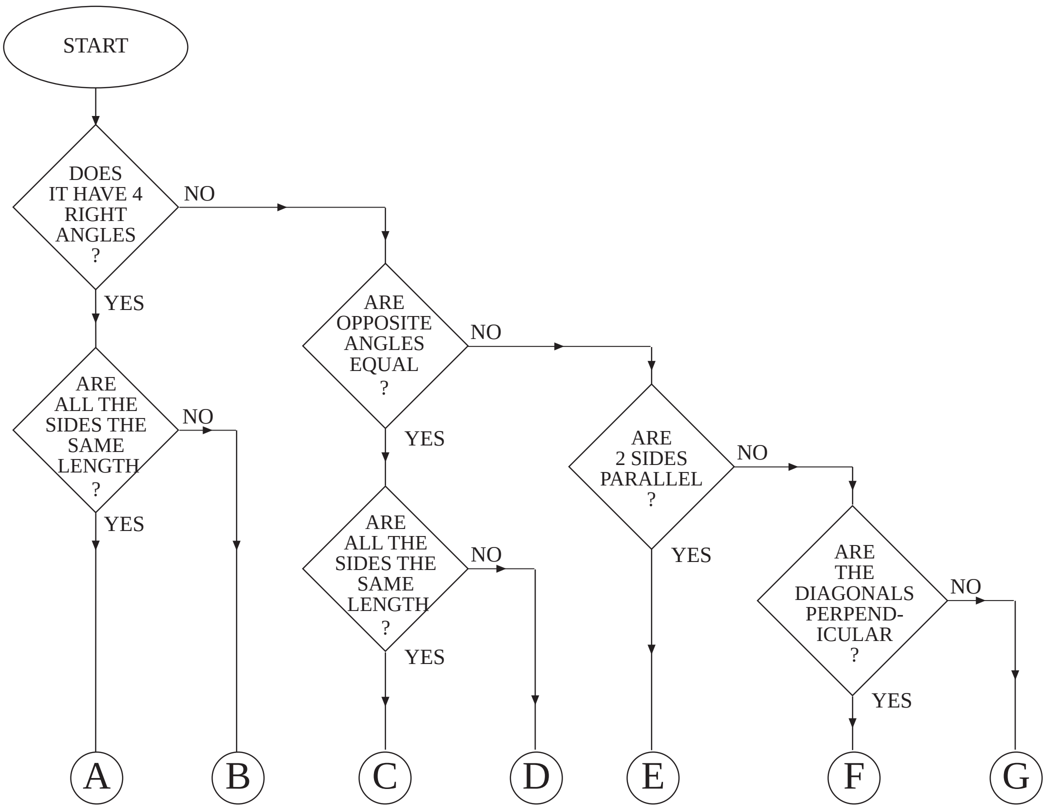
   (b) The quadrilateral could be a .... if ....

8. (a) Which quadrilaterals have *more than one* line of symmetry?

   (b) Draw diagrams to show them and their lines of symmetry.

   (c) Which quadrilaterals have rotational symmetry of order *greater than 1* ? List these quadrilaterals and state the order of their rotational symmetry.

9. The following flow chart is used to identify quadrilaterals:



Which type of quadrilateral arrives at each of the outputs, A to G ?

10. The following flow chart can be used to classify quadrilaterals, but some question boxes are empty. Copy and complete the flow chart.