10 Sequences

10.1 Constant Differences

In the first part of this unit we consider sequences where the difference between successive terms is the same every time. We also use formulae to create the terms of a sequence.

Example 1

Write down the next 3 terms of each of the following sequences:

(a) 7, 11, 15, 19, 23, ...

(b) 1, 9, 17, 25, 33, ...

Solution

(a) 7 11 15 19 23 ...

4 4 4 4

The difference between each term and the next is always 4. This value is called the *first difference*. So we can continue the sequence by adding 4 each time. This gives the sequence:

7, 11, 15, 19, 23, 27, 31, 35

(b) 1 9 17 25 33 ...

8 8 8 8

Here the difference between each term and the next is always 8. To continue the sequence we must keep on adding 8 every time. This gives the sequence:

1, 9, 17, 25, 33, 41, 49, 57

Example 2

A sequence is defined by the formula $u_n = 3n + 1$.

Calculate the first 5 terms of this sequence.

Solution

The first term, often called u_1, is formed by substituting $n = 1$ into the formula.

$$\begin{aligned} u_1 &= 3 \times 1 + 1 \\ &= 3 + 1 \\ &= 4 \end{aligned}$$

For the second term, substitute $n = 2$ to give:

$$\begin{aligned} u_2 &= 3 \times 2 + 1 \\ &= 7 \end{aligned}$$

For the third term, substitute $n = 3$ to give:

$$\begin{aligned} u_3 &= 3 \times 3 + 1 \\ &= 10 \end{aligned}$$

For the fourth term, substitute $n = 4$ to give:

$$\begin{aligned} u_4 &= 3 \times 4 + 1 \\ &= 13 \end{aligned}$$

For the fifth term, substitute $n = 5$ to give:

$$\begin{aligned} u_5 &= 3 \times 5 + 1 \\ &= 16 \end{aligned}$$

So the first 5 terms of the sequence are

4, 7, 10, 13, 16.

Example 3

The terms of a sequence are given by the formula $u_n = 8n - 3$.

Calculate:

(a) the first 3 terms of the sequence,

(b) the 100th term of the sequence,

(c) the 200th term of the sequence.

Solution

(a) $n = 1$ gives $u_1 = 8 \times 1 - 3$

$= 5$

$n = 2$ gives $u_2 = 8 \times 2 - 3$

$= 13$

$n = 3$ gives $u_3 = 8 \times 3 - 3$

$= 21$

So the first 3 terms are

5, 13, 21.

(b) $n = 100$ gives $u_{100} = 8 \times 100 - 3$

$= 797$

So the 100th term of the sequence is 797.

(c) $n = 200$ gives $u_{200} = 8 \times 200 - 3$

$= 1597$

So the 200th term of the sequence is 1597.

Exercises

1. Write down the next 3 terms of each of the following sequences:
   (a) 2, 5, 8, 11, 14, ...
   (b) 9, 18, 27, 36, 45, ...
   (c) 13, 14, 15, 16, 17, ...
   (d) 7, 15, 23, 31, 39, ...

2. Write down the next 3 terms of each of the following sequences:
   (a) 100, 98, 96, 94, 92, ...
   b) 20, 17, 14, 11, 8, ...
   (c) 48, 43, 38, 33, 28, ...
   (d) 17, 13, 9, 5, 1, ...

3. A sequence is defined by the formula $u_n = 6n - 2$.
   (a) Calculate the first 5 terms of the sequence.
   (b) What is the difference between the terms of the sequence?

4. A sequence is defined by the formula $u_n = 8n + 2$.
   (a) Calculate the first 5 terms of the sequence.
   (b) What is the difference between the terms of the sequence?
   (c) Write down the next 3 terms of the sequence.

5. A sequence is given by $u_n = 7n - 3$.
   (a) Calculate the first 4 terms of the sequence.
   (b) What is the difference between the terms of the sequence?
   (c) Explain where the difference appears in the formula for the terms.

6. A sequence is given by $u_n = 9n + 2$.

   (a) Calculate the first 4 terms of the sequence.

   (b) How does the difference between terms relate to the formula?

7. A sequence is given by the formula $u_n = 11n - 7$.

   (a) What would you expect to be the difference between the terms of the sequence?

   (b) Calculate the first 4 terms of the sequence and check your answer to part (a).

   (c) Calculate the 10th term of the sequence.

8. A sequence is defined by the formula $u_n = 82 - 4n$.

   (a) Calculate the first 5 terms of the sequence.

   (b) What is the difference between terms for the sequence?

   (c) How does this difference relate to the formula?

   (d) Calculate the 20th term of the sequence.

9. (a) Calculate the 100th term of the sequence given by $u_n = 8n - 5$.

   (b) Calculate the 25th term of the sequence given by $u_n = 11n - 3$.

   (c) Calculate the 200th term of the sequence given by $u_n = 3n + 22$.

   (d) Calculate the 58th term of the sequence defined by $u_n = 1000 - 5n$.

10. Four sequences, A, B, C and D, are defined by the following formulae:

    A $\quad u_n = 8n + 2$

    B $\quad u_n = 7n - 3$

    C $\quad u_n = 3n + 1$

    D $\quad u_n = 100 - 6n$

    (a) Which sequences have 4 as their first term?

    (b) Which sequence is *decreasing*?

    (c) Which sequence has a difference of 7 between terms?

    (d) Which sequence has 301 as its 100th term?

11. (a) Look at this part of a number line.
Write down the 2 missing numbers.

Copy and complete this sentence:

The numbers on this line go *up* in steps of ....

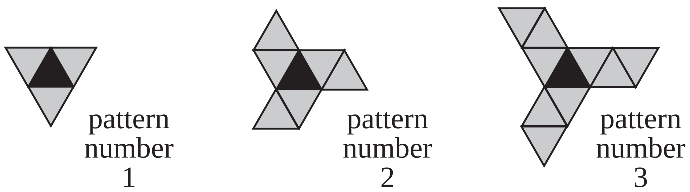
(b) This is a *different* number line.
Write down the 3 missing numbers.

Copy and complete this sentence:

The numbers on this line go *up* in steps of ....

(KS3/97/Ma/Tier 4-6/P1)

12. Jeff makes a sequence of patterns with black and grey triangular tiles.



The rule for finding the number of tiles in pattern number N in Jeff's sequence is:

$$\boxed{\text{number of tiles } = 1 + 3N}$$

(a) The 1 in this rule represents the *black tile.*

What does the $3N$ represent?

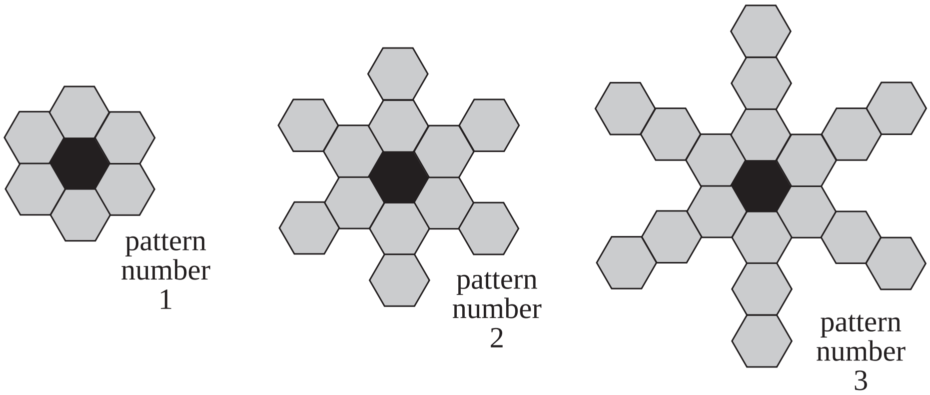
(b) Jeff makes *pattern number 12* in his sequence.

How many *black* tiles and how many *grey* tiles does he use?

(c) Jeff uses 61 tiles altogether to make a pattern in his sequence.

What is the number of the pattern he makes?

(d) Barbara makes a sequence of patterns with *hexagonal* tiles.



Each pattern in Barbara's sequence has 1 black tile in the middle.

Each new pattern has 6 more grey tiles than the pattern before.

Copy and complete the rule for finding the number of tiles in pattern number N in Barbara's sequence.

number of tiles = ... + ...

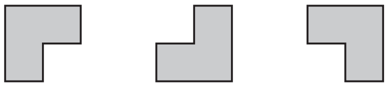
(e) Gwenno uses some tiles to make a *different* sequence of patterns.

The rule for finding the number of tiles in pattern number N in Gwenno's sequence is:

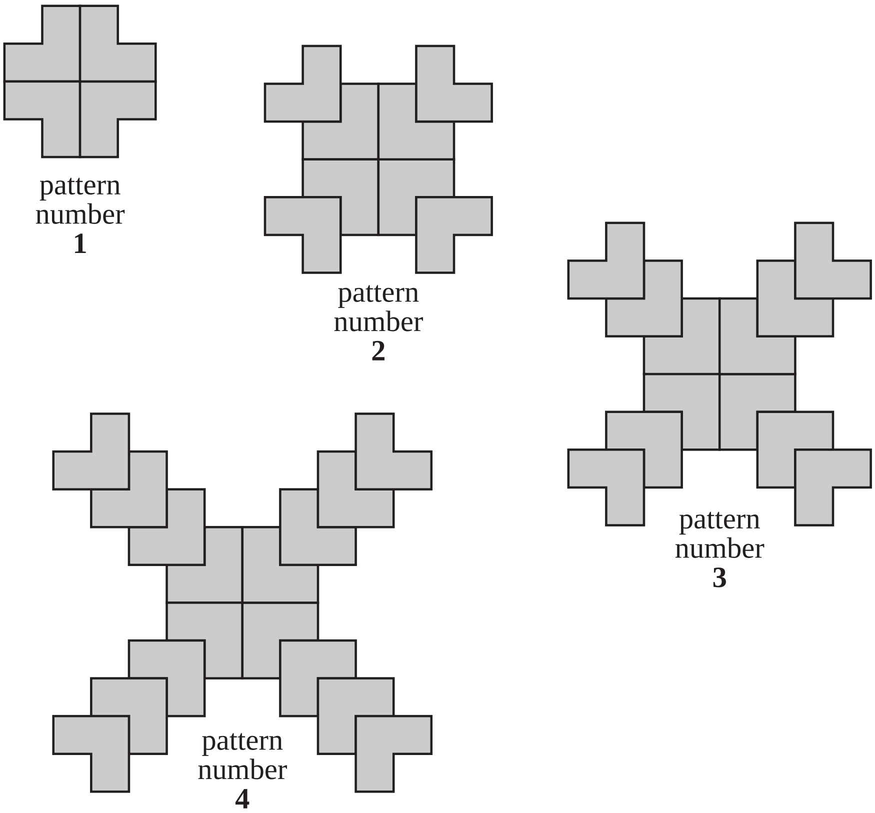
number of tiles $= 1 + 4N$

Draw what you think the first 3 patterns in Gwenno's sequence could be.

(KS3/99/Ma/Tier 5-7/P2)

13. Owen has some tiles like these: 

He uses the tiles to make a series of patterns.



(a) Each new pattern has *more tiles* than the one before.
The number of tiles goes up by the same amount each time.

How many *more* tiles does Owen add each time he makes a new pattern?

(b) *How many tiles* will Owen need altogether to make *pattern number* 6 ?

(c) *How many tiles* will Owen need altogether to make *pattern number* 9 ?

(d) Owen uses 40 tiles to make a pattern.

What is the number of the pattern he makes?

(KS3/98/Ma/Tier 4-6/P2)

10.2 Finding the Formula for a Linear Sequence

It is possible to determine a formula for linear sequences, i.e. sequences where the difference between successive terms is always the same.

The first differences for the number pattern

$$\begin{array}{lcccccccccccc} & 11 & & 14 & & 17 & & 20 & & 23 & & 26 & \ldots \\ \text{are} & & 3 & & 3 & & 3 & & 3 & & 3 & & \end{array}$$

If we look at the sequence $3n$, i.e. the multiples of 3, and compare it with our original sequence

our sequence	11	14	17	20	23	26
sequence $3n$	3	6	9	12	15	18

we can see easily that the formula that generates our number pattern is

nth term of sequence $= 3n + 8$

i.e. $u_n = 3n + 8$

If, however, we had started with the sequence

38 41 44 47 50 53 ...

the first differences would still have been 3 and the comparison of this sequence with the sequence $3n$

our sequence	38	41	44	47	50	53
sequence $3n$	3	6	9	12	15	18

would have led to the formula $u_n = 3n + 35$.

In the same way, the sequence

-7 -4 -1 2 5 8 ...

also has first differences 3 and the comparison

our sequence	-7	-4	-1	2	5	8
sequence $3n$	3	6	9	12	15	18

yields the formula $u_n = 3n - 10$.

From these examples, we can see that any sequence with constant first difference 3 has the formula

$u_n = 3n + c$

where the adjustment constant c may be either positive or negative.

This approach can be applied to any linear sequence, giving us the general rule that:

> If the *first* difference between *successive* terms is d, then
>
> $$u_n = d \times n + c$$

Example 1

Determine a formula for this sequence:

7, 13, 19, 25, 31, ...

Solution

First consider the differences between the terms,

$$\begin{array}{ccccccccccc} 7 & & 13 & & 19 & & 25 & & 31 & & \ldots \\ & 6 & & 6 & & 6 & & 6 & & & \end{array}$$

As the difference is always 6, we can write,

$$u_n = 6n + c$$

As the first term is 7, we can write down the equation:

$$\begin{aligned} 7 &= 6 \times 1 + c \\ &= 6 + c \\ c &= 1 \end{aligned}$$

So the formula will be,

$$u_n = 6n + 1$$

We can check that this formula is correct by testing it on other terms, for example,

$$\text{the 4th term } = 6 \times 4 + 1 = 25$$

which is correct.

Example 2

Determine a formula for this sequence:

2, 7, 12, 17, 22, 27, ...

Solution

First consider the differences between the terms,

$$\begin{array}{ccccccccccccc} 2 & & 7 & & 12 & & 17 & & 22 & & 27 & & \ldots \\ & 5 & & 5 & & 5 & & 5 & & 5 & & & \end{array}$$

The difference between each term is always 5, so the formula will be,

$$u_n = 5n + c$$

The first term can be used to form an equation to determine c:

$$2 = 5 \times 1 + c$$

$$2 = 5 + c$$

$$c = -3$$

So the formula will be,

$$u_n = 5n - 3$$

> Note that the constant term, c, is given by
>
> c = first term $-$ first difference

Example 3

Determine a formula for the sequence:

28, 25, 22, 19, 16, 13, ...

Solution

First consider the differences between the terms,

$$\begin{array}{ccccccccccc} 28 & & 25 & & 22 & & 19 & & 16 & & 13 \quad \ldots \\ & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & \\ & -3 & & -3 & & -3 & & -3 & & -3 & \end{array}$$

Here the difference is *negative* because the terms are becoming smaller.

Using the difference as -3 gives,

$$u_n = -3n + c$$

The first term is 28, so

$$28 = -3 \times 1 + c$$

$$28 = -3 + c$$

$$c = 31$$

The general formula is then,

$$u_n = -3n + 31$$

or

$$u_n = 31 - 3n$$

Exercises

1. For the sequence,

   7, 11, 15, 19, ...

   (a) calculate the *difference* between successive terms,

   (b) determine the *formula* that generates the sequence.

2. Determine the *formula* for each of the following sequences:

   (a) 6, 10, 14, 20, 24, ...

   (b) 11, 13, 15, 17, 19, ...

   (c) 9, 16, 23, 30, 37, ...

   (d) 34, 56, 78, 100, 122, ...

   (e) 22, 31, 40, 49, 58, ...

3. One number is missing from the following sequence:

   1, 6, 11, $\boxed{\quad}$, 21, 26, 31

   (a) What is the missing number?

   (b) Calculate the *difference* between successive terms.

   (c) Determine the *formula* that generates the sequence.

4. Determine the *general formula* for each of the following sequences:

   (a) 1, 4, 7, 10, 13, ...

   (b) 2, 6, 10, 14, 18, ...

   (c) 4, 13, 22, 31, 40, ...

   (d) 5, 15, 25, 35, 45, ...

   (e) 1, 20, 39, 58, 77, ...

5. For the sequence,

   18, 16, 14, 12, 10, ...

   (a) calculate the *difference* between successive terms,

   (b) determine the *formula* that generates the sequence.

6. Determine the *general formula* for each of the following sequences:

   (a) 19, 16, 13, 10, 7, ...

   (b) 100, 96, 92, 88, 84, ...

   (c) 41, 34, 27, 20, 13, ...

   (d) 66, 50, 34, 18, 2, ...

   (e) 90, 81, 72, 63, 54, ...

7. For the sequence,

   $-2, \quad -4, \quad -6, \quad -8, \quad -10, \quad -12, \quad ...$

   (a) calculate the *difference* between successive terms,

   (b) determine the *formula* for the sequence.

8. Determine the *formula* that generates each of the following sequences:

   (a) $0, \quad -5, \quad -10, \quad -15, \quad -20, \quad ...$

   (b) $-18, \quad -16, \quad -14, \quad -12, \quad -10, \quad ...$

   (c) $-5, \quad -8, \quad -11, \quad -14, \quad -17, \quad ...$

   (d) $8, \quad 1, \quad -6, \quad -13, \quad -20, \quad ...$

   (e) $-7, \quad -3, \quad 1, \quad 5, \quad 9, \quad ...$

9. A sequence has first term 20 and the difference between the terms is always 31.

   (a) Determine a *formula* to generate the terms of the sequence.

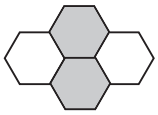
   (b) Calculate the *first* 5 *terms* of the sequence.

10. The second and third terms of a sequence are 16 and 27. The difference between successive terms in the sequence is always constant.

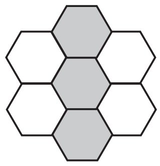
   (a) Determine the *general formula* for the sequence.

   (b) Calculate the *first* 5 *terms* of the sequence.

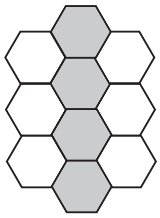
11. This is a series of patterns with grey and white tiles.



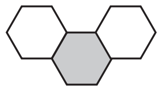
pattern number 1



pattern number 2



pattern number 3

The series of patterns continues by adding  each time.

   (a) Copy and complete this table:

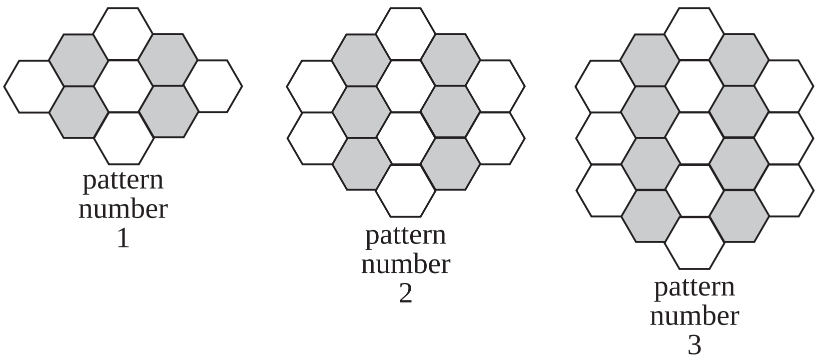
pattern number	number of *grey* tiles	number of *white* tiles
5		
16		

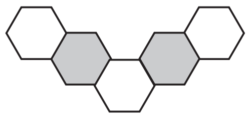
(b) Copy and complete this table by writing *expressions*:

pattern number	expression for the number of *grey* tiles	expression for the number of *white* tiles
n		

(c) Write an expression to show the *total* number of tiles in pattern number n. Simplify your expression.

(d) A different series of patterns is made with tiles.



The series of patterns continues by adding  each time.

For this series of patterns, write an expression to show the *total* number of tiles in pattern number n.

Show your working and *simplify* your expression.

(KS3/98/Ma/Tier 5-7/P1)

10.3 Second Differences and Quadratic Sequences

In section 10.2 we dealt with sequences where the differences between the terms was a constant value. In this section we extend this idea to sequences where the differences are *not constant*.

Example 1

(a) Calculate the first 6 terms of the sequence defined by the quadratic formula,

$$u_n = n^2 + n - 1$$

(b) Calculate the first differences between the terms.

(c) Comment on the results you obtain.

Solution

(a) Substituting $n = 1$ gives,

$$u_1 = 1^2 + 1 - 1$$
$$= 1$$

For $n = 2$,

$$u_2 = 2^2 + 2 - 1$$
$$= 5$$

For $n = 3$,

$$u_3 = 3^2 + 3 - 1$$
$$= 11$$

For $n = 4$,

$$u_4 = 4^2 + 4 - 1$$
$$= 19$$

For $n = 5$,

$$u_5 = 5^2 + 5 - 1$$
$$= 29$$

For $n = 6$,

$$u_6 = 6^2 + 6 - 1$$
$$= 41$$

So the first 6 terms are,

1, 5, 11, 19, 29, 41

(b) The differences can now be calculated,

$$\begin{array}{ccccccccccc} 1 & & 5 & & 11 & & 19 & & 29 & & 41 \\ & 4 & & 6 & & 8 & & 10 & & 12 & \end{array}$$

(c) Note that the differences between the first differences are constant. They are all equal to 2. These are called the *second differences,* as shown below.

$$\begin{array}{lccccccccccc} \textit{Sequence} & 1 & & 5 & & 11 & & 19 & & 29 & & 41 \\ \textit{First differences} & & 4 & & 6 & & 8 & & 10 & & 12 & \\ \textit{Second differences} & & & 2 & & 2 & & 2 & & 2 & & \end{array}$$

Example 2

(a) Calculate the first 5 terms of the sequence defined by the quadratic formula

$u_n = 3n^2 - n - 2$

(b) Determine the first and second differences for this sequence.

(c) Comment on your results.

Solution

(a) For $n = 1$,

$$\begin{aligned} u_1 &= 3 \times 1^2 - 1 - 2 \\ &= 3 - 1 - 2 \\ &= 0 \end{aligned}$$

For $n = 2$,

$$\begin{aligned} u_2 &= 3 \times 2^2 - 2 - 2 \\ &= 8 \end{aligned}$$

For $n = 3$,

$$\begin{aligned} u_3 &= 3 \times 3^2 - 3 - 2 \\ &= 22 \end{aligned}$$

For $n = 4$,

$$\begin{aligned} u_4 &= 3 \times 4^2 - 4 - 2 \\ &= 42 \end{aligned}$$

For $n = 5$,

$$\begin{aligned} u_5 &= 3 \times 5^2 - 5 - 2 \\ &= 68 \end{aligned}$$

So the sequence is,

0, 8, 22, 42, 68, ...

(b) The differences are calculated below:

$$\begin{array}{lccccccccccc} \textit{Sequence} & 0 & & 8 & & 22 & & 42 & & 68 & & \ldots \\ & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & & \\ \textit{First differences} & & 8 & & 14 & & 20 & & 26 & & & \\ & & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & & & \\ \textit{Second differences} & & & 6 & & 6 & & 6 & & & & \end{array}$$

(c) Again, the second differences are constant; this time they are all 6.

Note

For a sequence defined by a *quadratic formula*, the second differences will be constant and equal to twice the number of n^2.

For example,

$u_n = n^2 + n - 1$ Second difference = 2

$u_n = 3n^2 - n - 2$ Second difference = 6

$u_n = 5n^2 - n + 7$ Second difference = 10

Example 3

Determine a formula for the general term of the sequence,

2, 9, 20, 35, 54, ...

Solution

Consider the first and second differences of the sequence:

2 9 20 35 54 ...

7 11 15 19

4 4 4

As the second differences are constant and equal to 4, the formula will begin

$$u_n = 2n^2 + \dots$$

To determine the rest of the formula, subtract $2n^2$ from each term of the sequence, as shown below:

Sequence	2	9	20	35	54	...
$2n^2$	2	8	18	32	50	
New sequence	0	1	2	3	4	

1 1 1 1

The new sequence has a constant difference of 1 and begins with 0, so for this sequence the formula is $n - 1$.

Combining this with the $2n^2$ gives

$$u_n = 2n^2 + n - 1$$

Example 4

(a) Calculate the first and second differences for the sequence,

4, 1, 0, 1, 4, 9, ...

(b) Use the differences to determine the next 2 terms of the sequence.

(c) Determine a formula for the general term of the sequence.

Solution

(a) 4 1 0 1 4 9 ...

-3 -1 1 3 5

2 2 2 2

(b) Extending the sequences above gives,

$$\begin{array}{ccccccccccccccc} 4 & & 1 & & 0 & & 1 & & 4 & & 9 & & 16 & & 25 \quad \ldots \\ & -3 & & -1 & & 1 & & 3 & & 5 & & 7 & & 9 & \\ & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & \end{array}$$

(c) As the second differences are constant and all equal to 2, the formula will contain an 'n^2' term, and be of the form

$$u_n = n^2 + an + b$$

We must now determine the values of a and b. The easiest way to do this is to subtract n^2 from each term of the sequence, to form a new, simpler sequence.

our sequence	4	1	0	1	4	9
sequence n^2	1	4	9	16	25	36
new sequence	3	– 3	– 9	– 15	– 21	– 27

The new sequence

$$\begin{array}{ccccccccccc} 3 & & -3 & & -9 & & -15 & & -21 & & -27 \quad \ldots \\ & -6 & & -6 & & -6 & & -6 & & -6 & \end{array}$$

has constant first differences of -6 so will be given by $-6n + b$.

Using the first term gives,

$$3 = -6 \times 1 + b$$
$$b = 9$$

Thus the formula for the simpler sequence is $-6n + 9$.

Now combining this with the n^2 term gives,

$$u_n = n^2 - 6n + 9$$

Exercises

1. (a) Calculate the first 6 terms of the sequence defined by,

$$u_n = n^2 + 2n + 1$$

   (b) Calculate the second differences for the sequence.

   (c) Use the differences to calculate the next 2 terms of the sequence.

2. A sequence has its general term defined as,

$$u_n = 8n^2 - n - 1$$

   (a) What would you expect to be the second differences for the sequence?

   (b) Calculate the first 5 terms of the sequence.

   (c) Calculate the second differences for the sequence. Did you obtain the values you expected?

3. A sequence is listed below:

   6, 9, 14, 21, 30, 41, ...

   (a) Calculate the second differences for the sequence.

   (b) Determine the formula for the general term of the sequence.

4. Determine the formula for the general term of each of the following sequences:

   (a) 1, 7, 17, 31, 49, 71, ...

   (b) 6, 18, 38, 66, 102, 146, ...

   (c) – 5, 10, 35, 70, 115, 170, ...

   (d) 1, 10, 25, 46, 73, 106, ...

5. A sequence is listed below:

   2, 9, 20, 35, 54, 77, ...

   (a) Calculate the second differences for this sequence.

   (b) Form a simpler sequence by subtracting $2n^2$ from each term.

   (c) Determine a formula for the general term of the simpler sequence.

   (d) Determine a formula for the general term of the original sequence.

6. (a) Calculate the second differences of the sequence,

   6, 17, 36, 63, 98, 141, ...

   (b) Determine the formula for the general term of the sequence.

7. Determine the formula for the general term of each of the following sequences:

   (a) 3, 17, 39, 69, 107, ...

   (b) 5, 18, 37, 62, 93, ...

   (c) 9, 23, 45, 75, 113, ...

   (d) – 4, 12, 38, 74, 120, ...

8. (a) Calculate the second differences for the sequence,

   9, 4, – 5, – 18, – 35, ...

   (b) Determine the formula for the general term of the sequence.

   (c) Hence show that the 20th term of the sequence is –770.

9. Determine the formula for the general term of the sequence,

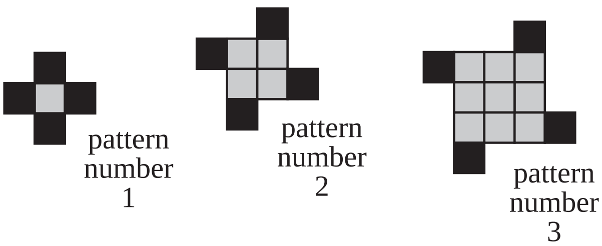
   6, 10, 12, 12, 10, 6, ...

10. (a) Calculate the first, second and third differences for the sequence,

   6, 13, 32, 69, 130, 221, ...

   (b) Determine a formula for the general term of the sequence.

11. This is a series of patterns with grey and black tiles.



   (a) How many grey tiles and black tiles will there be in pattern number 8 ?

   (b) How many grey tiles and black tiles will there be in pattern number 16 ?

   (c) How many grey tiles and black tiles will there be in pattern number P ?

   (d) T = total number of grey tiles and black tiles in a pattern
   P = pattern number

   Use symbols to write down an equation connecting T and P.

(Ks3/96/Ma/Tier 6-8/P2)

10.4 Special Sequences

Before going on to look at harder examples, we list some of the important sequences that you should know:

1, 3, 5, 7, 9, 11, 13, ...
the *odd* numbers $u_n = 2n - 1$

2, 4, 6, 8, 10, 12, 14, ...
the *even* numbers $u_n = 2n$

1, 4, 9, 15, 25, 36, 49, ...
the *square* numbers $u_n = n^2$

1, 8, 27, 64, 125, 216, 343, ...
the *cube* numbers $u_n = n^3$

1, 3, 6, 10, 15, 21, 28, ...
the *triangular* numbers $u_n = \frac{1}{2}n(n+1)$

There is one other important sequence, namely the *prime* numbers

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

Note: there is no formula for calculating the *n*th prime number.

We now look at other, harder sequences generated by algebraic rules.

Example 1

(a) Write down the next 3 terms of the sequence,

1, 1, 2, 3, 5, 8, 13, ...

(b) Determine a formula for calculating the *n*th term.

Solution

(a) Use the first differences to extend the sequence:

1 1 2 3 5 8 13 ...
0 1 1 2 3 5

Note that the first differences, ignoring the first 0, are in fact the actual sequence itself. These can then be used to extend the sequence:

1 1 2 3 5 8 13 21 34 55 ...
0 1 1 2 3 5 8 13 21

(b) Each term is the sum of the two previous terms; for example,

$$u_3 = u_2 + u_1$$

$$u_4 = u_3 + u_2$$

We can express this mathematically as,

$$u_n = u_{n-1} + u_{n-2}$$

This formula connects u_n to the two previous terms, rather than n which we used in the earlier sections. This sequence is actually a special sequence and is called the *Fibonacci* sequence.

Example 2

The first two terms of a sequence are 1 and 2. The sequence is defined as,

$$u_n = 2u_{n-1} + u_{n-2}$$

Calculate the next 3 terms of the sequence.

Solution

Note that $u_1 = 1$ and $u_2 = 2$.

$$\begin{aligned} u_3 &= 2u_2 + u_1 \\ &= 2 \times 2 + 1 \\ &= 5 \end{aligned}$$

$$\begin{aligned} u_4 &= 2u_3 + u_2 \\ &= 2 \times 5 + 2 \\ &= 12 \end{aligned}$$

$$\begin{aligned} u_5 &= 2u_4 + u_3 \\ &= 2 \times 12 + 5 \\ &= 29 \end{aligned}$$

So the first 5 terms of the sequence are,

1, 2, 5, 12, 29

Example 3

For the sequence,

6, 12, 24, 48, 96, ...

(a) calculate the next 2 terms of the sequence,

(b) determine a general formula for the *n*th term.

Solution

(a) Note that, in this sequence, each term is twice the previous term.

$$6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \xrightarrow{\times 2} 48 \xrightarrow{\times 2} 96 \xrightarrow{\times 2} 192 \xrightarrow{\times 2} 384 \quad \ldots$$

(b) Consider how each term is formed:

$$u_1 = 6 = 3 \times 2$$

$$u_2 = 12 = 3 \times 2 \times 2 = 3 \times 2^2$$

$$u_3 = 24 = 3 \times 2 \times 2 \times 2 = 3 \times 2^3$$

$$u_4 = 48 = 3 \times 2 \times 2 \times 2 \times 2 = 3 \times 2^4$$

Hence the general term will be $u_n = 3 \times 2^n$.

This sequence is an example of an *exponential* sequence.

Example 4

Consider the sequence,

$$\frac{3}{5}, \quad \frac{7}{8}, \quad \frac{11}{11}, \quad \frac{15}{14}, \quad \frac{19}{17}, \quad \frac{23}{20}, \quad \ldots$$

(a) Write down the next 2 terms of the sequence,

(b) Determine the general formula for the *n*th term of the sequence.

Solution

(a) It is best to consider the numerators and the denominators separately.

First consider the sequence of numerators,

$$\begin{array}{ccccccccccccc} 3 & & 7 & & 11 & & 15 & & 19 & & 23 & & \ldots \\ & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & & \\ & 4 & & 4 & & 4 & & 4 & & 4 & & & \end{array}$$

As the difference between the terms is 4, we have

$$u_n = 4n + a$$

and using the first term,

$$3 = 4 \times 1 + a$$

$$a = -1$$

Hence

$$u_n = 4n - 1$$

Now consider the sequence of denominators,

$$\begin{array}{ccccccccccccc} 5 & & 8 & & 11 & & 14 & & 17 & & 20 & & \ldots \\ & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & \diagdown\diagup & & & \\ & 3 & & 3 & & 3 & & 3 & & 3 & & & \end{array}$$

As the differences between terms is 3, we have

$$u_n = 3n + b$$

and, using the first term,

$$5 = 3 \times 1 + b$$

$$b = 2$$

Hence

$$u_n = 3n + 2$$

So for the given sequence of fractions we have,

$$u_n = \frac{4n - 1}{3n + 2}$$

Example 5

What happens to the sequence defined by,

$$u_n = \frac{n - 1}{n + 1}$$

as n becomes larger and larger?

Solution

The following table lists n and u_n for several values of n.

From the table it can be seen that the values of $u_n = \dfrac{n-1}{n+1}$ get larger and larger as n increases.

However, the numerator is always smaller than the denominator, so each value u_n must be smaller than 1.

It follows that, as n gets larger and larger, the values of u_n must get closer and closer to 1.

n	$u_n = \frac{n-1}{n+1}$	u_n *to 3 decimal places*
1	0	0
2	$\frac{1}{3}$	0.333
3	$\frac{2}{4}$	0.5
4	$\frac{3}{5}$	0.6
5	$\frac{4}{6}$	0.667
10	$\frac{9}{11}$	0.818
20	$\frac{19}{21}$	0.905
50	$\frac{49}{51}$	0.961
100	$\frac{99}{101}$	0.980
500	$\frac{499}{501}$	0.996
1000	$\frac{999}{1001}$	0.998
2000	$\frac{1999}{2001}$	0.999

Exercises

1. Calculate the next three terms in each of the following sequences:

   (a) 1, 3, 4, 7, 11, 18, ...

   (b) 4, 9, 13, 22, 35, ...

   (c) $\frac{1}{2}$, $\frac{2}{5}$, $\frac{3}{8}$, $\frac{4}{11}$, $\frac{5}{14}$, ...

   (d) 5, 15, 45, 135, 405, ...

2. Calculate the first 6 terms of each of the following sequences:

(a) $u_1 = 0, \quad u_2 = 3 \quad , \quad u_n = u_{n-1} + u_{n-2}$

(b) $u_1 = 3, \quad u_2 = 4 \quad , \quad u_n = 2u_{n-1} + u_{n-2}$

(c) $u_1 = 6, \quad u_2 = 10 \quad , \quad u_n = 3u_{n-1} - u_{n-2}$

(d) $u_1 = 1, \quad u_2 = 2 \quad , \quad u_n = u_{n-1} \times u_{n-2}$

3. (a) Calculate the next 3 terms of the sequence,

1, 4, 16, 64, 256, ...

(b) Determine a formula for the *n*th term of the sequence.

4. Determine the formula for the general term of each of the following sequences:

(a) 15, 75, 375, 1875, 9375, ...

(b) 1, 3, 9, 27, 81, ...

(c) 20, 200, 2000, 20 000, 200 000, ...

(d) 4, 28, 196, 1372, 9604, ...

5. (a) Determine the general formula for the terms of the sequence,

1, 7, 13, 19, 25, 31, ...

(b) Determine the general formula for the terms of the sequence,

2, 10, 18, 26, 34, 42, ...

(c) Determine the general formula for the terms of the sequence,

$\frac{1}{2}, \quad \frac{7}{10}, \quad \frac{13}{18}, \quad \frac{19}{26}, \quad \frac{25}{34}, \quad \frac{31}{42}, \quad ...$

6. Determine the general formula for the terms of each of the following sequences:

(a) $\frac{1}{4}, \quad \frac{2}{5}, \quad \frac{3}{6}, \quad \frac{4}{7}, \quad \frac{5}{8}, \quad ...$

(b) $\frac{1}{3}, \quad \frac{3}{11}, \quad \frac{5}{19}, \quad \frac{7}{27}, \quad \frac{9}{35}, \quad ...$

(c) $\frac{5}{7}, \quad \frac{14}{12}, \quad \frac{23}{17}, \quad \frac{32}{22}, \quad \frac{41}{27}, \quad ...$

(d) $\frac{1}{10}, \quad \frac{7}{20}, \quad \frac{13}{40}, \quad \frac{19}{80}, \quad \frac{25}{160}, \quad ...$

7. Determine the formula for the general term of each of the following sequences, and also calculate the 10th term of each sequence.

(a) $1, \ \frac{6}{7}, \ \frac{9}{11}, \ \frac{4}{5}, \ \frac{15}{19}, \ \ldots$

(b) $\frac{1}{5}, \ \frac{3}{4} \ \ 1, \ \frac{8}{7}, \ \frac{21}{17}, \ \ldots$

8. (a) Complete the following table for $u_n = \frac{2n}{n+1}$.

n	u_n	u_n *to 3 decimal places*
1		
5		
10		
50		
100		
500		
1000		
2000		

(b) Describe what happens to u_n as n becomes larger and larger.

9. Complete tables similar to the one in question 8, for each of the following sequences:

(a) $u_n = \frac{5n-1}{n}$

(b) $u_n = \frac{6n+1}{n}$

(c) $u_n = \frac{3n+1}{n+1}$

(d) $u_n = \frac{n+1}{2n}$

Comment on the results you obtain.

10. What do you think will happen to each of the sequences below as n becomes large?

(a) $u_n = \frac{4n}{n+1}$

(b) $u_n = \frac{7n+1}{n}$

(c) $u_n = \frac{n}{2n+1}$

(d) $u_n = \frac{4n}{2n-1}$

Test your predictions with some larger and larger values of n.

11. (a) Here is a number chain:

$$2 \rightarrow 4 \rightarrow 6 \rightarrow 8 \rightarrow 10 \rightarrow 12 \rightarrow$$

The rule is: *add on* 2 *each time*

A *different* number chain is:

$$2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow 32 \rightarrow 64 \rightarrow$$

What could the rule be?

(b) Some number chains start like this:

$$1 \rightarrow 5 \rightarrow$$

Write down three *different* ways to continue this number chain.

For each chain write down the next three numbers.
Then write down the rule you are using.

(KS3/97/Ma/Tier 3-5/P2)

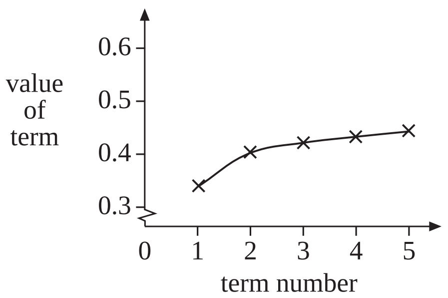
12. Each term of a number sequence is made by adding 1 to the numerator and 2 to the denominator of the previous term.

Here is the beginning of the number sequence:

$$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots$$

(a) Write an expression for the nth term of the sequence.

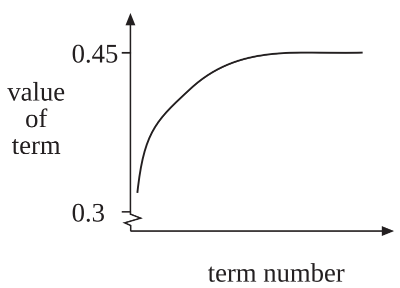
(b) The first five terms of the sequence are shown on the graph.



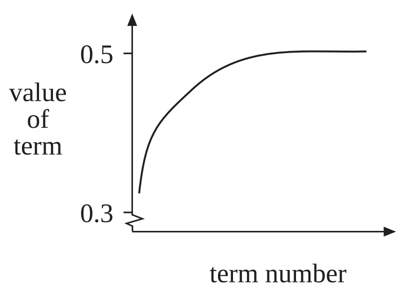
The sequence goes on and on for ever.

Which of the following four graphs shows how the sequence continues?

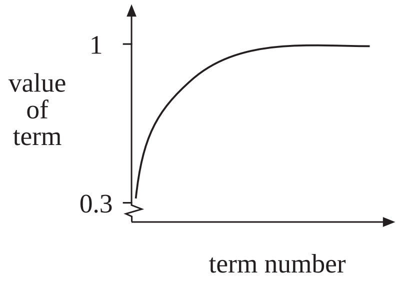
Graph 1



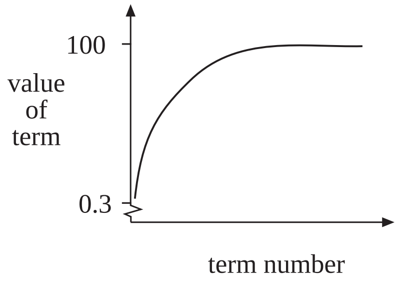
Graph 2



Graph 3



Graph 4



(c) The nth term of a different sequence is $\dfrac{n}{n^2+1}$.

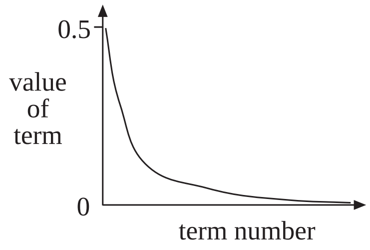
The *first term* of the sequence is $\dfrac{1}{2}$.

Write down the next *three terms.*

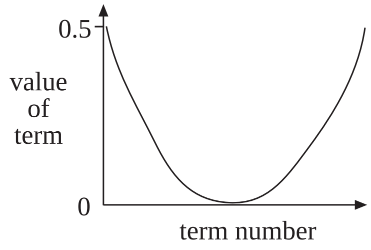
(d) This new sequence also goes on and on for ever.

Which of the four graphs below shows how the sequence continues?

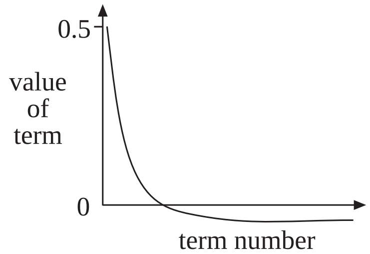
Graph 1



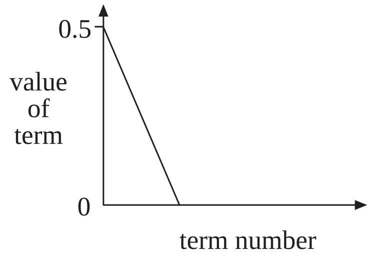
Graph 2



Graph 3



Graph 4



(KS3/99/Ma/Tier 6-8/P1)