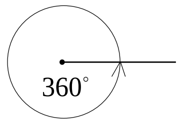
3 Angle Geometry

3.1 Measuring Angles

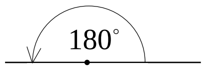
A *protractor* can be used to measure or draw angles.

Note

The angle around a complete circle is is 360°.

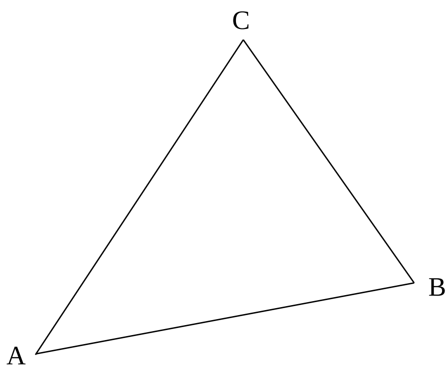


The angle around a point on a straight line is 180°.



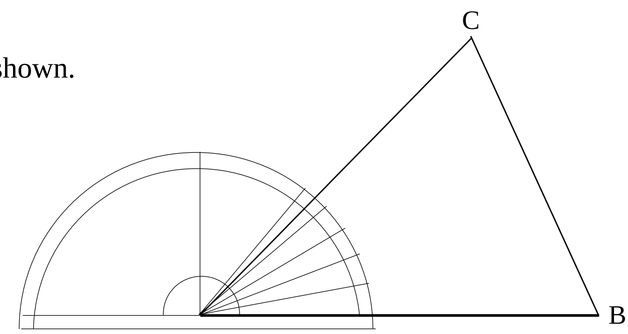
Worked Example 1

Measure the angle CAB in the triangle shown.



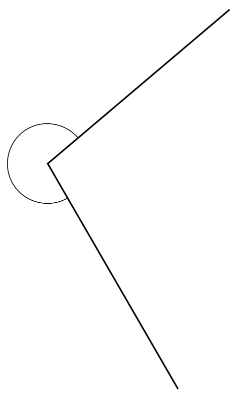
Solution

Place a protractor on the triangle as shown.
The angle is measured as 47°.

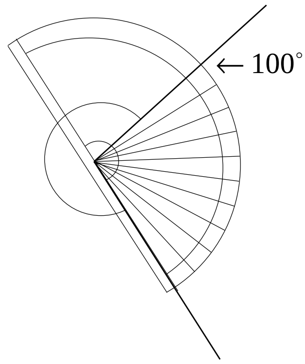


Worked Example 2

Measure this angle.



Solution



Using a protractor, the smaller angle is measured as 100°.

So

$$\text{required angle} = 360^\circ - 100^\circ$$
$$= 260^\circ$$

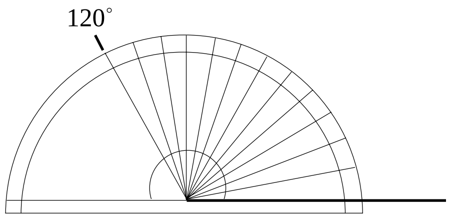
Worked Example 3

Draw angles of

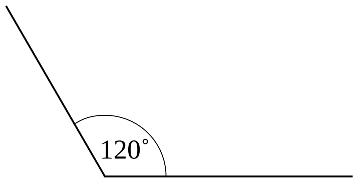
(a) 120° (b) 330°.

Solution

(a) Draw a horizontal line.
Place a protractor on top of the line and draw a mark at 120°.



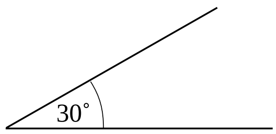
Then remove the protractor and draw the angle.



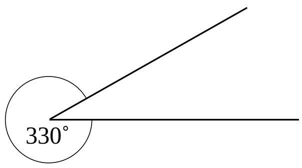
(b) To draw the angle of 330°, first subtract 330° from 360°:

$$360^\circ - 330^\circ = 30^\circ$$

Draw an angle of 30°.

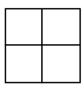


The larger angle will be 330°.



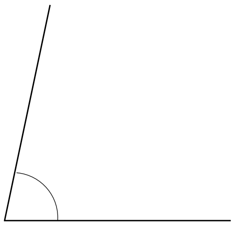
Just for Fun

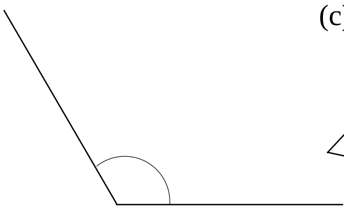
Arrange 12 toothpicks as shown.
Remove only 2 toothpicks so as to leave only 2 squares.

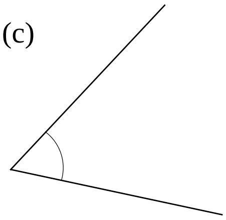


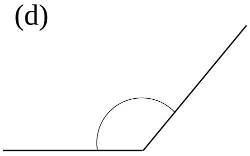
Exercises

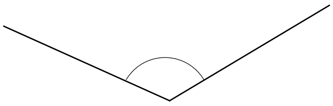
1. Estimate the size of each angle, then measure it with a protractor.

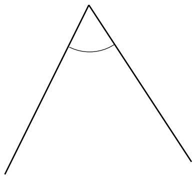
(a)


(b)


(c)


(d)


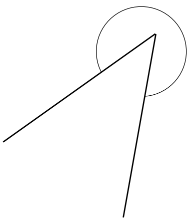
(e)


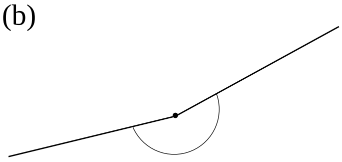
(f)


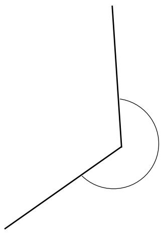
2. Draw angles with the following sizes.

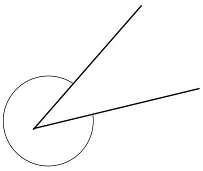
(a)	50°	(b)	70°	(c)	82°
(d)	42°	(e)	80°	(f)	100°
(g)	140°	(h)	175°	(i)	160°

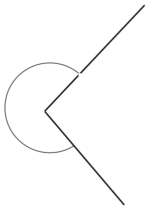
3. Measure these angles.

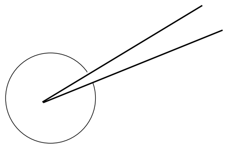
(a)


(b)


(c)


(d)


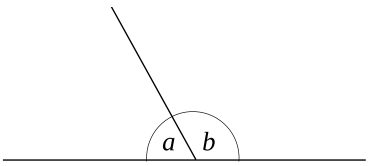
(e)


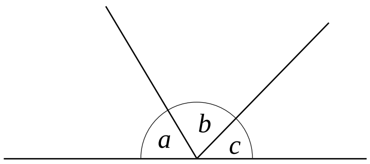
(f)


4. Draw angles with the following sizes.

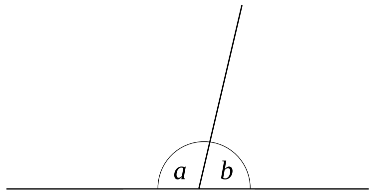
(a)	320°	(b)	190°	(c)	260°
(d)	210°	(e)	345°	(f)	318°

5. Measure each named (a, b, c) angle below and add up the angles in each diagram. What do you notice?

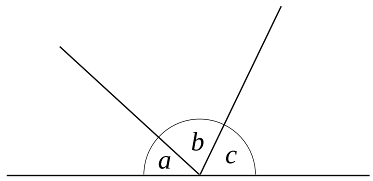
(a)


(b)


(c)

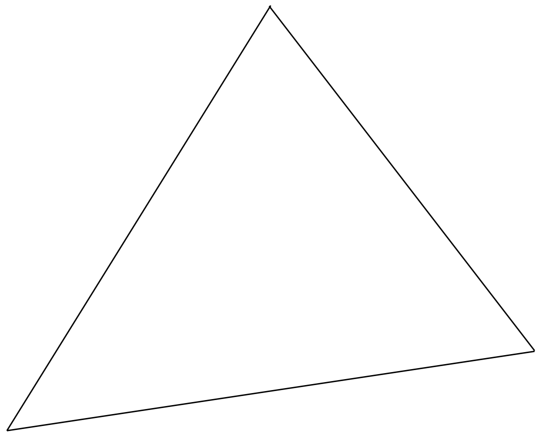


(d)

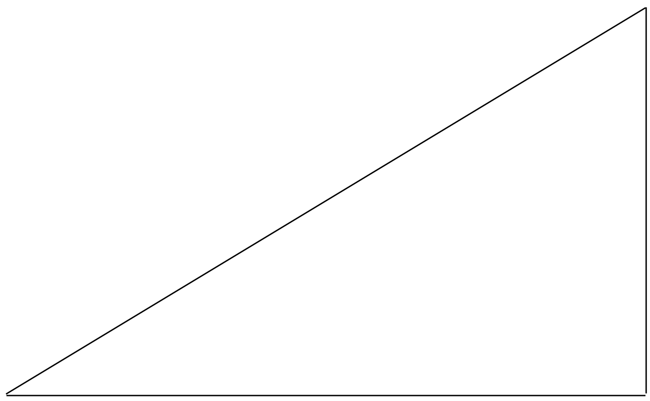


6. For each triangle below, measure each angle and add up the three angles you obtain.

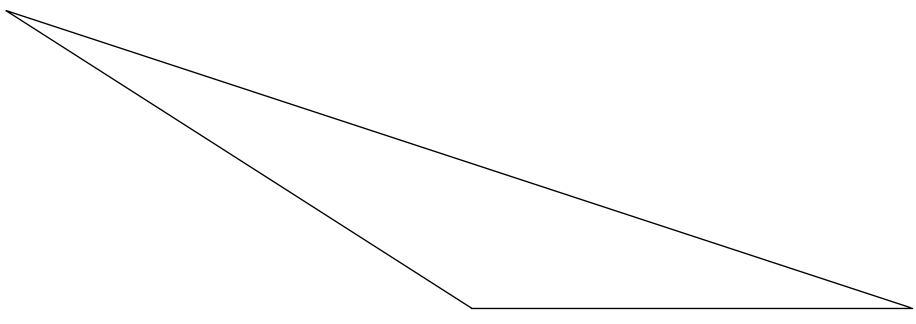
(a)



(b)

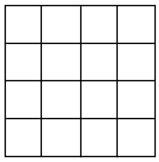


(c)

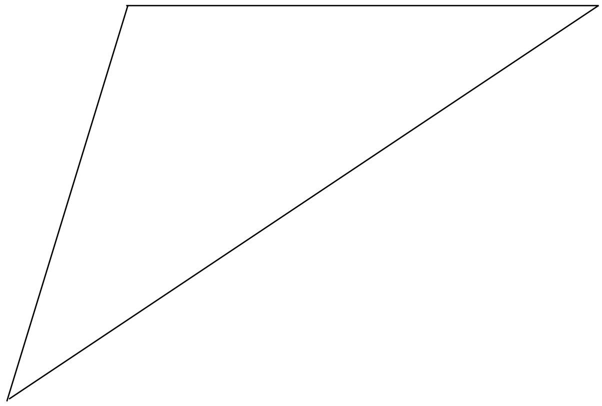


Investigation

How many squares are there in the given figure?



(d)

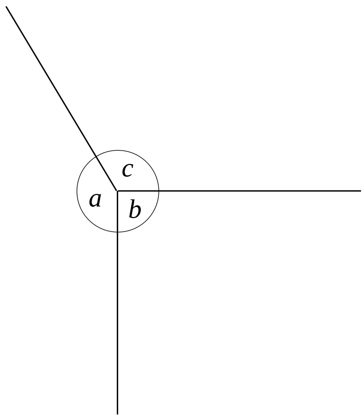


Do you obtain the same final result in each case?

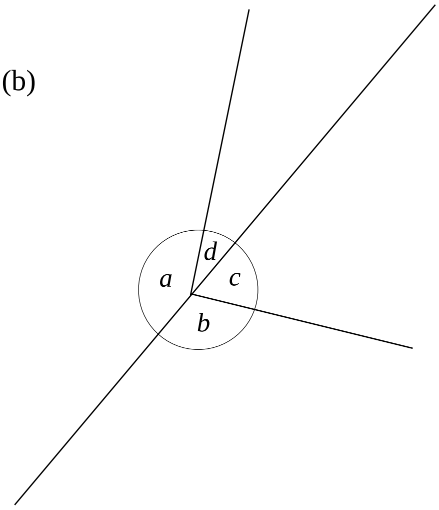
7. In each diagram below, measure the angles marked with letters and find their total.

What do you notice about the totals?

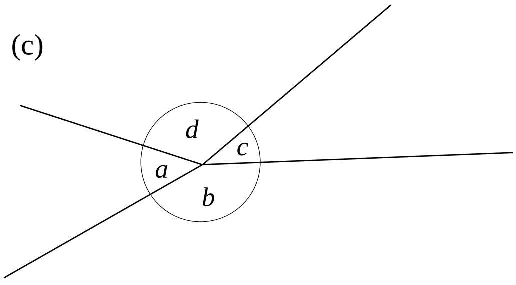
(a)



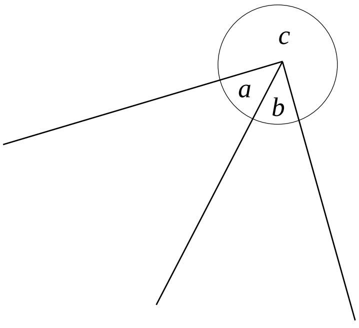
(b)



(c)

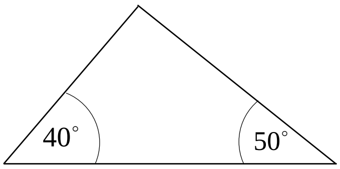


(d)



8. (a) Draw a straight line that is 10 cm long.

(b) Draw angles of 40° and 50° at each end to form the triangle shown in the diagram.

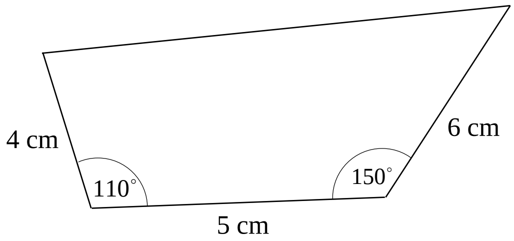


(c) Measure the lengths of the other two sides and the size of the other angle.

9. The diagram shows a rough sketch of a quadrilateral.

   (a) Draw the quadrilateral accurately.

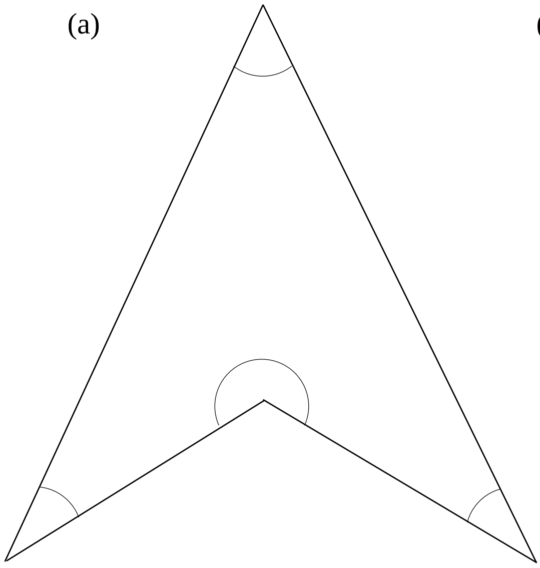
   (b) Measure the length of the fourth side and the size of the other two angles.



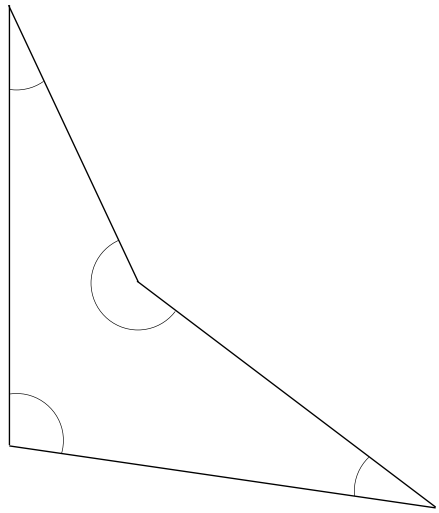
10. Measure the *interior* (inside) angles of these quadrilaterals.

    In each case find the total sum of the angles. What do you notice?

(a)



(b)



11. Draw two different pentagons.

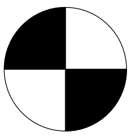
    (a) Measure each of the angles in both pentagons.

    (b) Add up your answers to find the total of the angles in each pentagon.

    (c) Do you think that the angles in a pentagon will always add up to the same number?

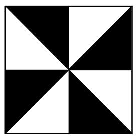
3.2 Line and Rotational Symmetry

An object has *rotational symmetry* if it can be rotated about a point so that it fits on top of itself without completing a full turn. The shapes below have rotational symmetry.



In a complete turn this shape fits on top of itself two times.

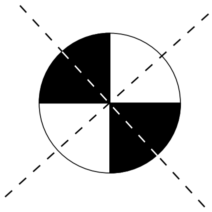
It has rotational symmetry of order 2.



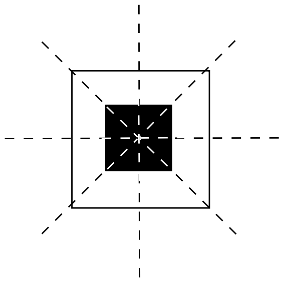
In a complete turn this shape fits on top of itself four times.

It has rotational symmetry of order 4.

Shapes have *line symmetry* if a mirror could be placed so that one side is an exact reflection of the other. These imaginary 'mirror lines' are shown by dotted lines in the diagrams below.



This shape has
2 lines of symmetry.



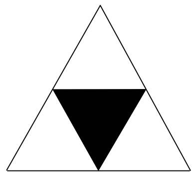
This shape has
4 lines of symmetry.

Worked Example 1

For the shape opposite state:

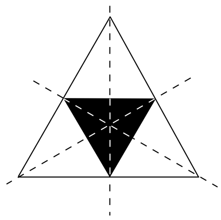
(a) the number of lines of symmetry,

(b) the order of rotational symmetry.

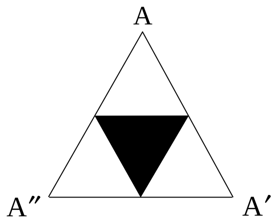


Solution

(a) There are 3 lines of symmetry as shown.



(b) There is rotational symmetry with order 3, because the point marked A could be rotated to A' then to A" and fit exactly over its original shape at each of these points.

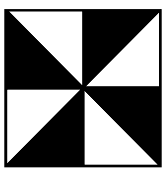


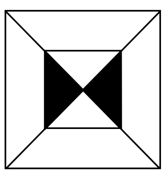
Exercises

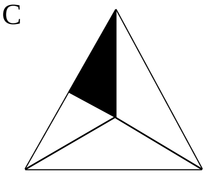
1. Which of the shapes below have

(a) line symmetry (b) rotational symmetry?

For line symmetry, copy the shape and draw in the mirror lines.
For rotational symmetry state the order.

A


B


C


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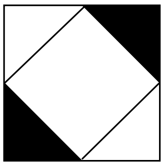
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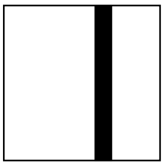
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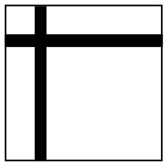
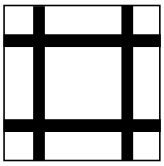


G

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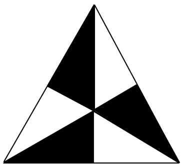
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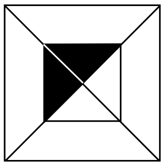
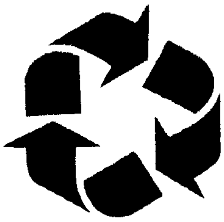


2. For each shape below state:

   (a) whether the shape has any symmetry;

   (b) how many lines of symmetry it has;

   (c) the order of symmetry if it has rotational symmetry.

A

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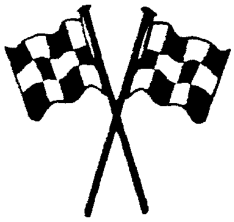
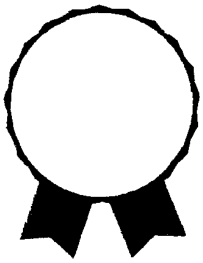

C


D

E

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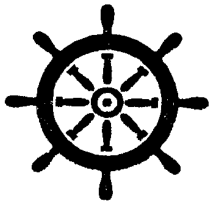

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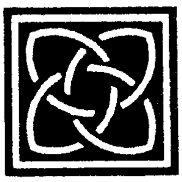
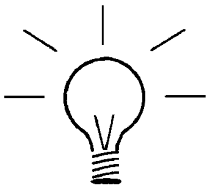
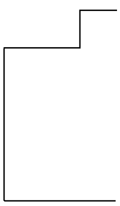
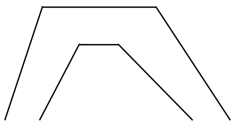
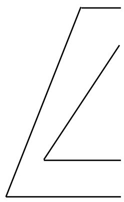
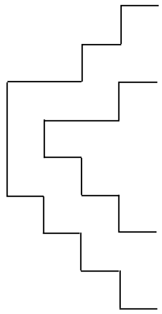
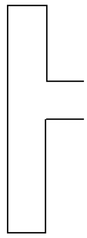
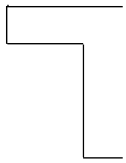
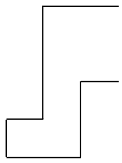
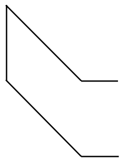
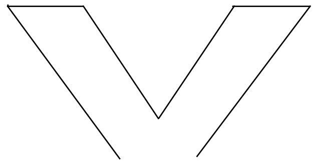

I


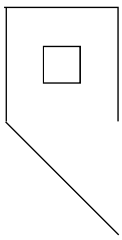
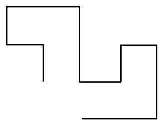
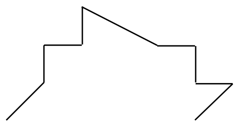
3. Copy and complete each shape below so that it has line symmetry but not rotational symmetry. Mark clearly the lines of symmetry.

(a)

(b)

(c)


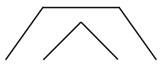
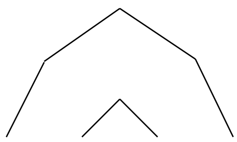
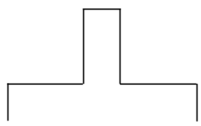
(d)  (e)  (f) 

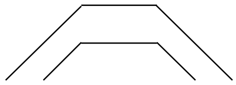
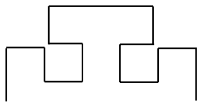
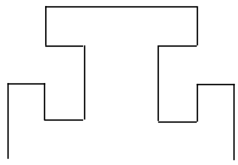
4. Copy and, if possible, complete each shape below, so that they have rotational symmetry, but not line symmetry. In each case state the order of the rotational symmetry.

(a)  (b)  (c) 

(d)  (e)  (f) 

5. Copy and complete each of the following shapes, so that they have both rotational and line symmetry. In each case draw the lines of symmetry and state the order of the rotational symmetry.

(a)  (b)  (c) 

(d)  (e)  (f) 

6. Draw a square and show all its lines of symmetry.

7. (a) Draw a triangle with:

   (i) 1 line of symmetry (ii) 3 lines of symmetry.

   (b) Is it possible to draw a triangle with 2 lines of symmetry?

8. Draw a shape which has 4 lines of symmetry.

9. Draw a shape with rotational symmetry of order:

   (a) 2 (b) 3 (c) 4 (d) 5

10. Can you draw:

   (a) a pentagon with exactly 2 lines of symmetry,

   (b) a hexagon with exactly 2 lines of symmetry,

   (c) an octagon with exactly 3 lines of symmetry?

11. These are the initials of the International Association of Whistlers.

   I A W

   Which of these letters has rotational symmetry?

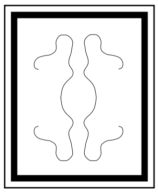
   (SEG)

12. Which of the designs below have line symmetry?

   (a)

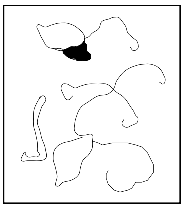
Taj Mahal floor tile

   (b)



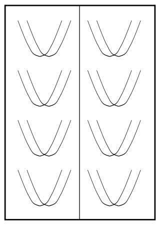
Asian carpet design

   (c)



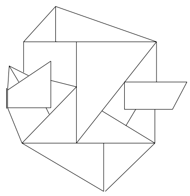
Contemporary art

   (d)



Wallpaper pattern

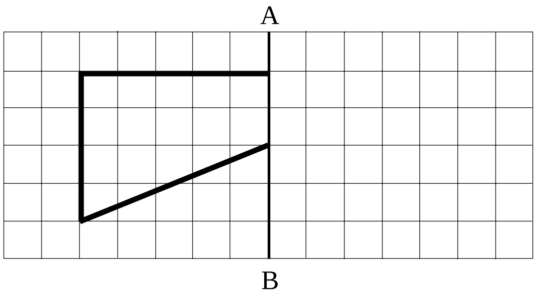
   (e)



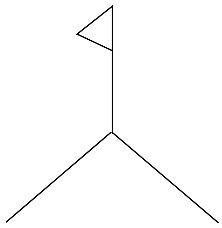
Tile design

   (LON)

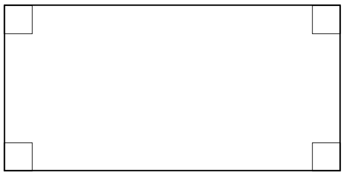
13. (a) Copy and draw the reflection of this shape in the mirror line AB.



(b) Copy and complete the diagram so that it has rotational symmetry.



(c) What is the order of rotational symmetry of this shape?



(SEG)

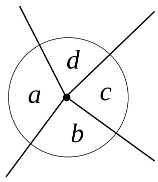
3.3 Angle Geometry

There are a number of important results concerning angles in different shapes, at a point and on a line. In this section the following results will be used.

1. *Angles at a Point*

   The angles at a point will always add up to 360°.

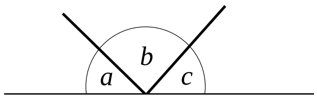
   It does not matter how many angles are formed at the point – their total will always be 360°.



$a + b + c + d = 360°$

2. *Angles on a Line*

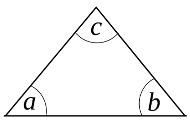
   Any angles that form a straight line add up to 180°.



$a + b + c = 180°$

3. *Angles in a Triangle*

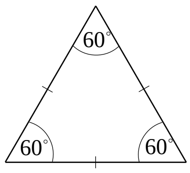
   The angles in any triangle add up to 180°.



$a + b + c = 180°$

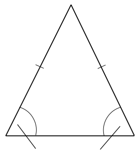
4. *Angles in an Equilateral Triangle*

   In an equilateral triangle all the angles are 60° and all the sides are the same length.



5. *Angles in an Isosceles Triangle*

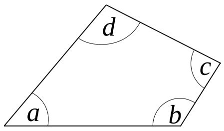
   In an isosceles triangle two sides are the same length and the two base angles are the same size.



equal angles

6. *Angles in a quadrilateral*

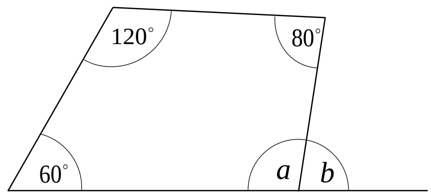
   The angles in any quadrilateral add up to 360°.



$a + b + c + d = 360°$

Worked Example 1

Find the sizes of angles a and b in the diagram below.



Solution

First consider the quadrilateral. All the angles of this shape must add up to $360°$, so

$$\begin{aligned} 60° + 120° + 80° + a &= 360° \\ 260° + a &= 360° \\ a &= 360° - 260° \\ &= 100° \end{aligned}$$

Then consider the straight line formed by the angles a and b. These two angles must add up to $180°$ so,

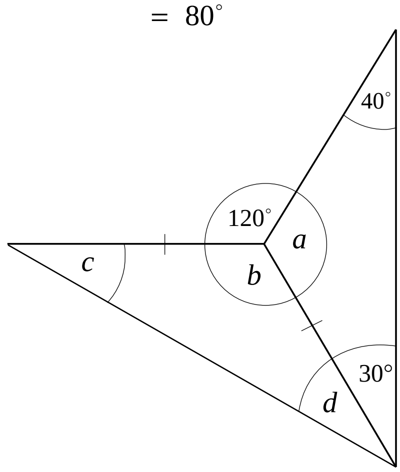
$$a + b = 180°$$

but $a = 100°$, so

$$\begin{aligned} 100° + b &= 180° \\ b &= 180° - 100° \\ &= 80° \end{aligned}$$

Worked Example 2

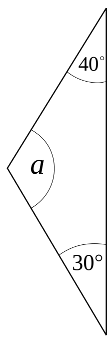
Find the angles a, b, c and d in the diagram.



Solution

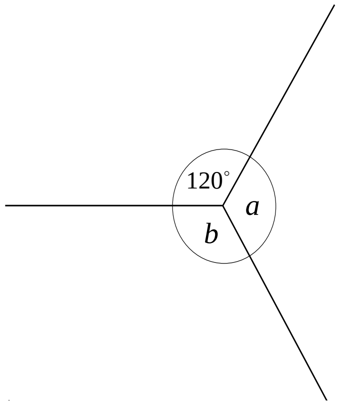
First consider the triangle shown.

The angles of this triangle must add up to $180°$,



So,
$$\begin{aligned} 40° + 30° + a &= 180° \\ 70° + a &= 180° \\ a &= 110° \end{aligned}$$

Next consider the angles round the point shown.



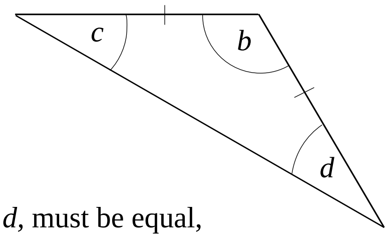
The three angles must add up to 360°, so

$$120^\circ + b + a = 360^\circ$$

but $a = 110^\circ$, so

$$\begin{aligned} 120^\circ + 110^\circ + b &= 360^\circ \\ 230^\circ + b &= 360^\circ \\ b &= 360^\circ - 230^\circ \\ &= 130^\circ. \end{aligned}$$

Finally, consider the second triangle.



The angles must add up to 180°, so

$$c + b + d = 180^\circ$$

As this is an isosceles triangle the two base angles, c and d, must be equal, so using $c = d$ and the fact that $b = 130^\circ$, gives

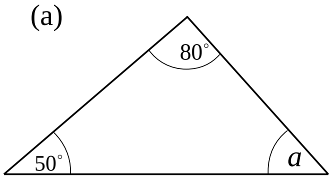
$$\begin{aligned} c + 130^\circ + c &= 180^\circ \\ 2c &= 180^\circ - 130^\circ \\ &= 50^\circ \\ c &= 25^\circ \end{aligned}$$

As $c = 25^\circ$, $d = 25^\circ$.

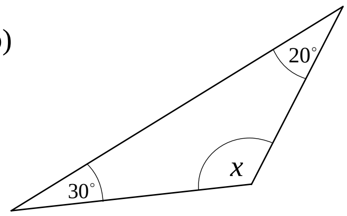
Exercises

1. Find the size of the angles marked with a letter in each diagram.

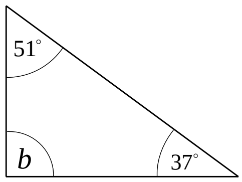
(a)



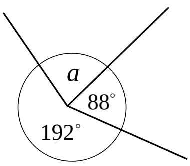
(b)



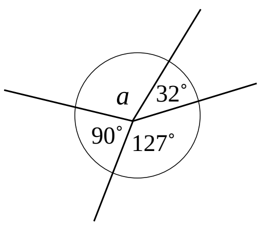
(c)



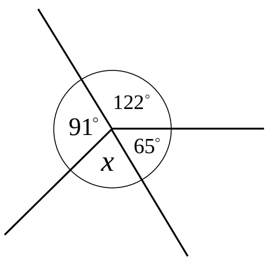
(d)



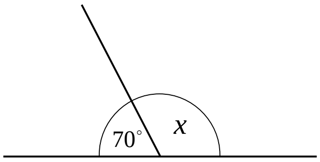
(e)



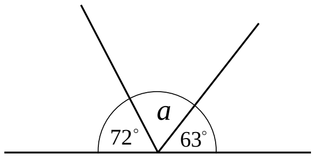
(f)



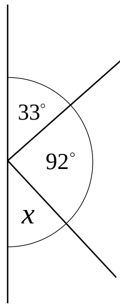
(g)



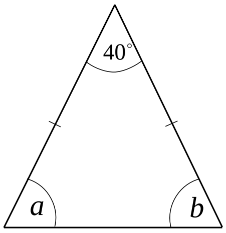
(h)



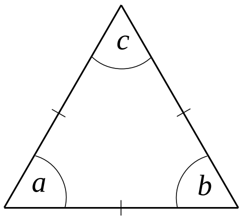
(i)



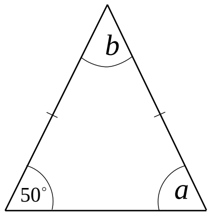
(j)



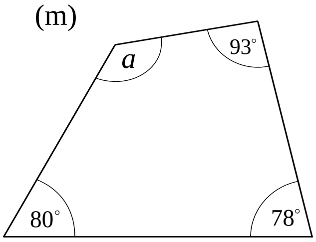
(k)



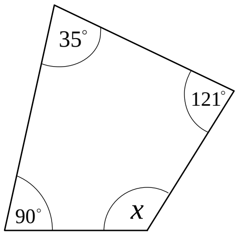
(l)



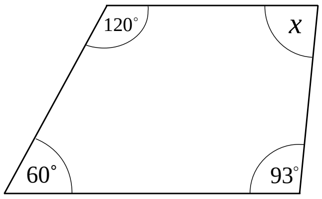
(m)



(n)

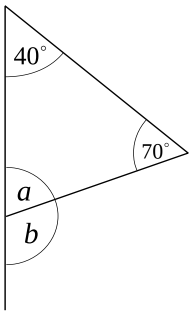


(o)

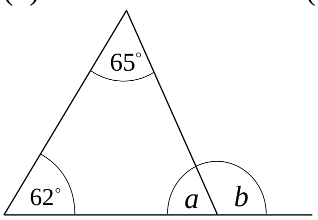


2. (a) For each triangle, find the angles marked *a* and *b* .

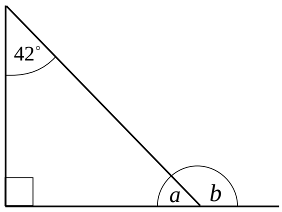
(i)



(ii)



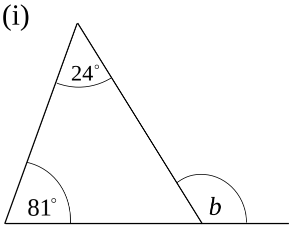
(iii)



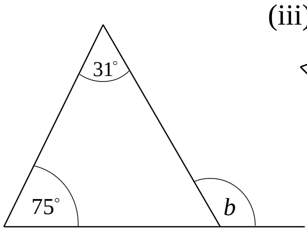
(b) What do you notice about the angle marked *b* and the other two angles given in each problem?

(c) Find the size of the angle *b* in each problem below without working out the size of any other angles.

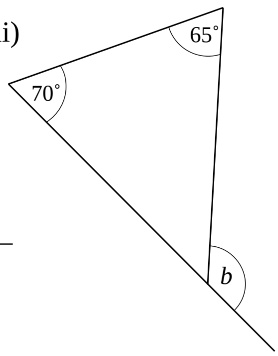
(i)



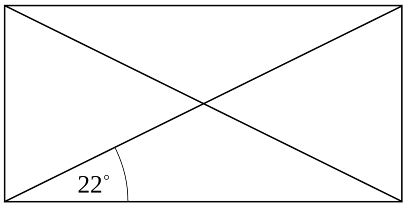
(ii)



(iii)



3. The diagram below shows a rectangle with its diagonals drawn in.

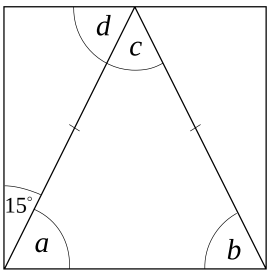


(a) Copy the diagram and mark in all the other angles that are 22°.

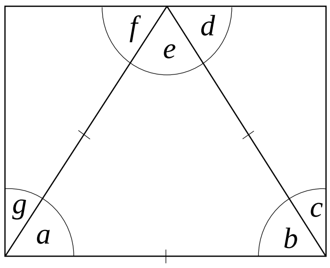
(b) Find the sizes of all the other angles.

4. Find the angles marked with letters in each of the following diagrams.
In each diagram the lines all lie inside a rectangle.

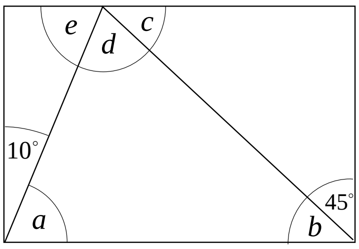
(a)



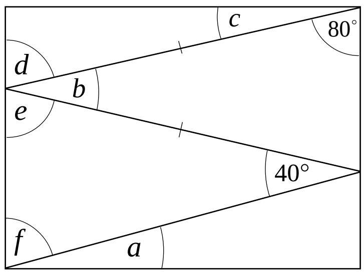
(b)



(c)

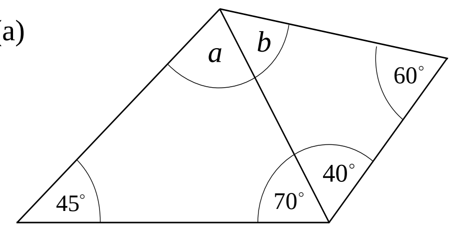


(d)

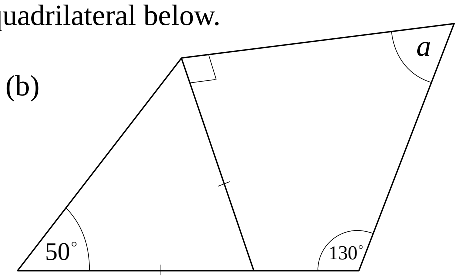


5. Find the angles marked with letters in each quadrilateral below.

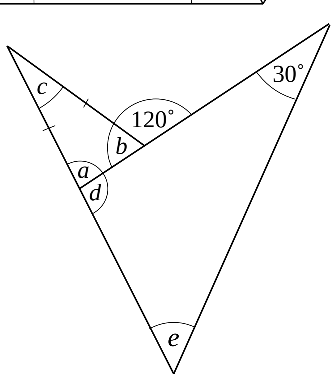
(a)



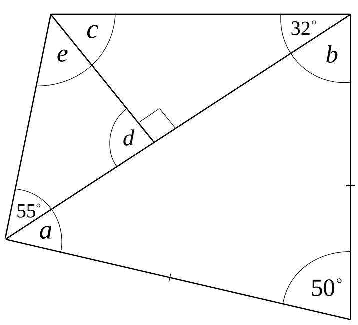
(b)



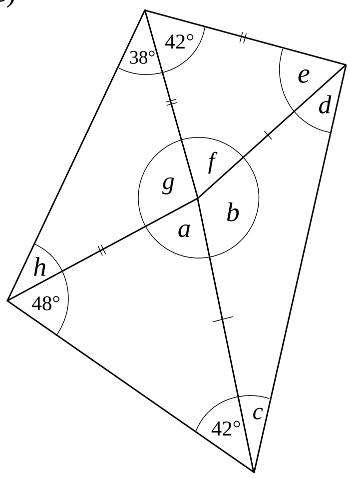
(c)



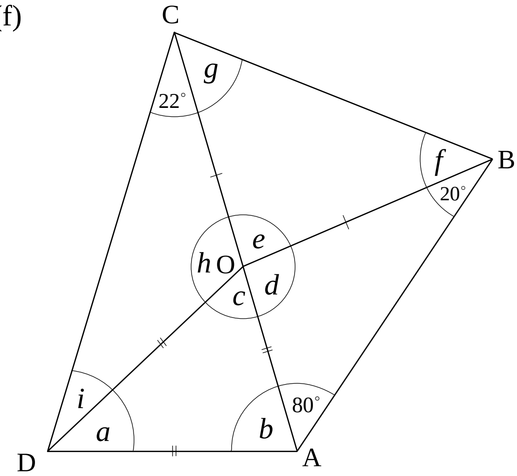
(d)



(e)

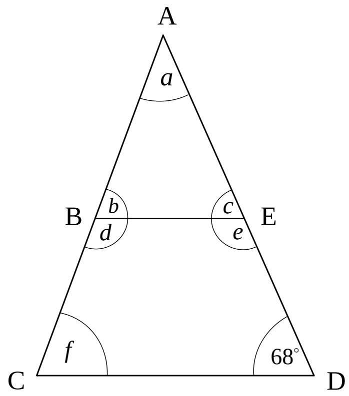


(f)



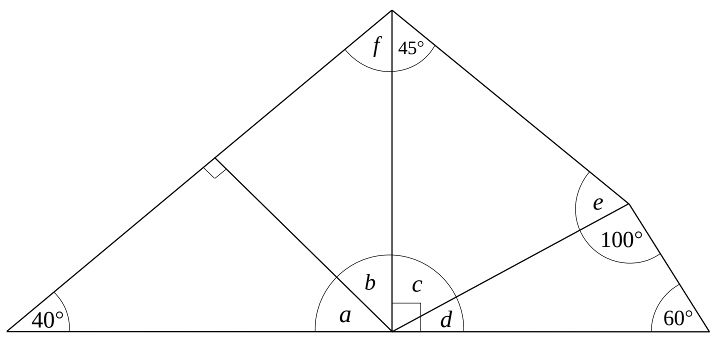
AOC is a straight line.

6. A swing is built from two metal frames as shown below.



The lengths of AB and AE are the same and the lengths of AC and AD are the same. Find the sizes of the angles *a*, *b*, *c*, *d*, *e* and *f*.

7. The diagram shows a wooden frame that forms part of the roof of a house.



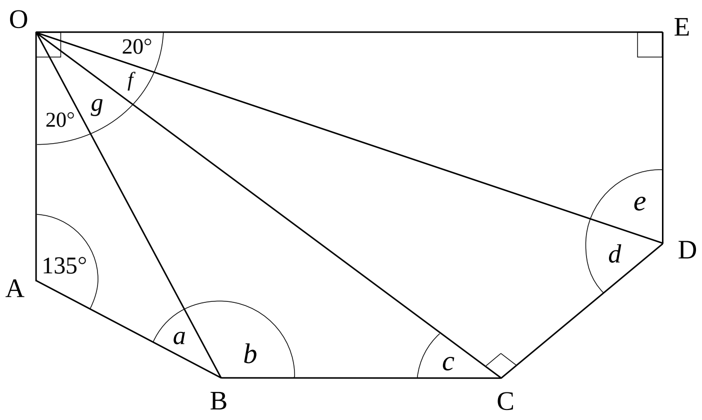
Find the sizes of the angles *a*, *b*, *c*, *d*, *e* and *f*.

Information

The word 'geometry' is derived from the Greek words, ge (earth) and metrein (to measure). Euclid's masterpiece, 'The Elements', survived as the basic textbook for over 2 000 years. The geometry we are studying in this unit is sometimes referred to as Euclidean geometry.

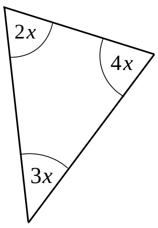
8. The diagram shows the plan for a conservatory. Lines are drawn from the point O to each of the other corners. Find all the angles marked with letters, if

$$A\hat{B}C = B\hat{C}D = C\hat{D}E = 135^\circ .$$

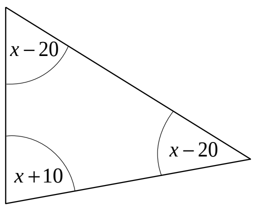


9. Write down an equation and use it to find the value of x in each diagram.

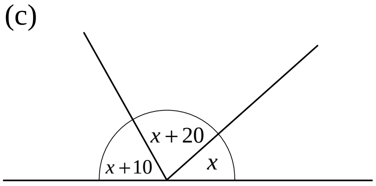
(a)



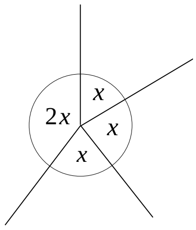
(b)



(c)



(d)



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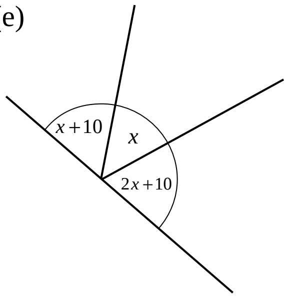

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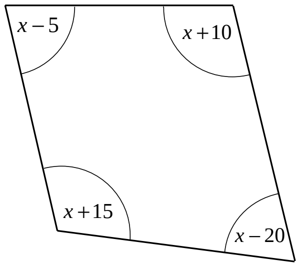
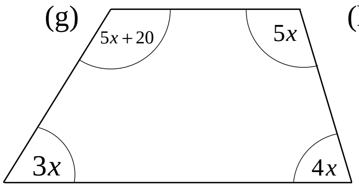
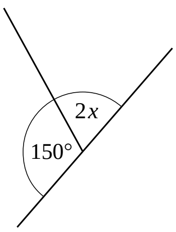


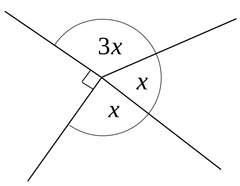
(g)



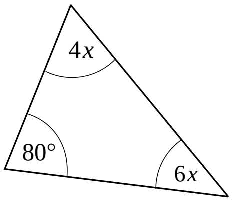
(h)



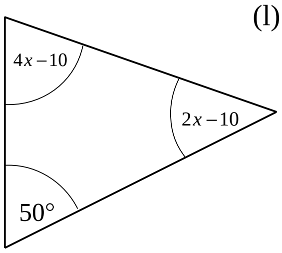
(i)



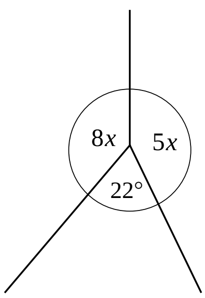
(j)



(k)



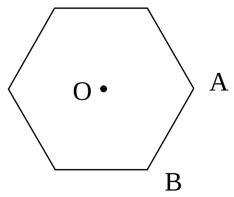
(l)



10. The diagram shows a regular hexagon.

    O is the point at the centre of the hexagon.

    A and B are two vertices.



    (a) Write down the order of rotational symmetry of the regular hexagon.

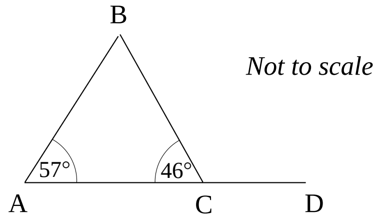
    (b) Draw the lines from O to A and from O to B.

        (i) Write down the size of angle AOB.

        (ii) Write down the mathematical name for triangle AOB.

(LON)

11. Calculate angles BCD and ABC, giving reasons for your answers.



(MEG)

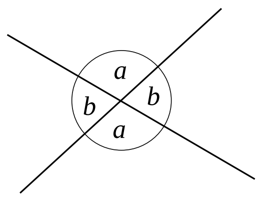
3.4 Angles with Parallel and Intersecting Lines

Opposite Angles

When any two lines cross, two pairs of equal angles are formed.

The two angles marked *a* are a pair of *opposite* equal angles.

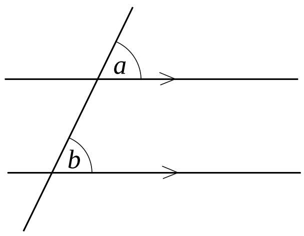
The angles marked *b* are also a pair of opposite equal angles.



Corresponding Angles

When a line crosses a pair of parallel lines, $a = b$.

The angles *a* and *b* are called *corresponding* angles.



Alternate Angles

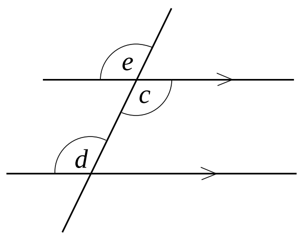
The angles *c* and *d* are equal.

Proof

This result follows since *c* and *e* are opposite angles, so $c = e$, and *e* and *d* are corresponding angles, so $c = d$.

Hence $c = e = d$

The angles *c* and *d* are called *alternate* angles.



Supplementary Angles

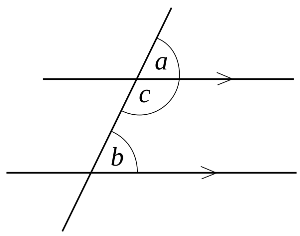
The angles *b* and *c* add up to $180°$.

Proof

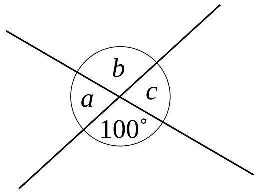
This result follows since $a + c = 180°$ (straight line), and $a = b$ since they are corresponding angles.

Hence $b + c = 180°$.

These angles are called *supplementary* angles.



Worked Example 1



Find the angles marked a, b and c.

Solution

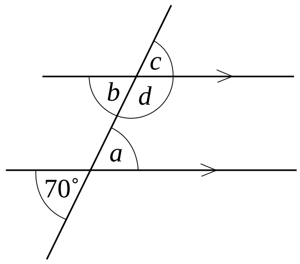
There are two pairs of opposite angles here so:

$$b = 100 \text{ and } a = c.$$

Also a and b form a straight line so

$$\begin{aligned} a + b &= 180° \\ a + 100° &= 180° \\ a &= 80°, \text{ so } c = 80°. \end{aligned}$$

Worked Example 2



Find the sizes of the angles marked a, b, c and d in the diagram.

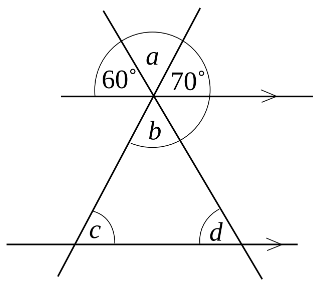
Solution

First note the two parallel lines marked with arrow heads.

Then find a. The angle a and the angle marked $70°$ are *opposite* angles, so $a = 70°$.
The angles a and b are *alternate* angles so $a = b = 70°$.
The angles b and c are *opposite* angles so $b = c = 70°$.
The angles a and d are a pair of *interior* angles, so $a + d = 180°$, but $a = 70°$, so

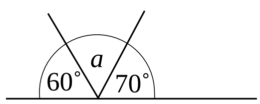
$$\begin{aligned} 70° + d &= 180° \\ d &= 180° - 70° \\ &= 110°. \end{aligned}$$

Worked Example 3



Find the angles marked a, b, c and d in the diagram.

Solution



To find the angle a, consider the three angles that form a straight line. So

$$\begin{aligned} 60° + a + 70° &= 180° \\ a &= 180° - 130° \\ &= 50°. \end{aligned}$$

The angle marked b is opposite the angle a, so $b = a = 50°$.

Now c and d can be found using *corresponding* angles.

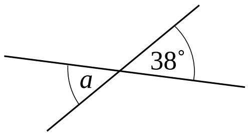
The angle c and the $70°$ angle are corresponding angles, so $c = 70°$.

The angle d and the $60°$ angle are corresponding angles, so $d = 60°$.

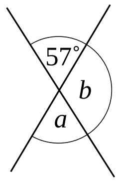
Exercises

1. Find the angles marked in each diagram, *giving reasons for your answers.*

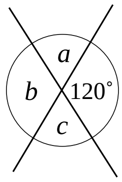
(a)



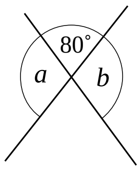
(b)



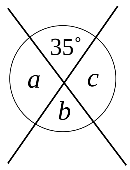
(c)



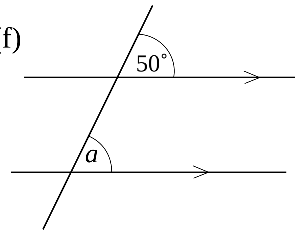
(d)



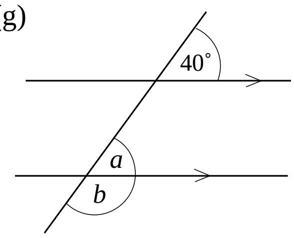
(e)



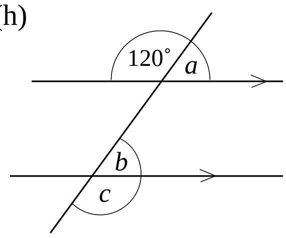
(f)



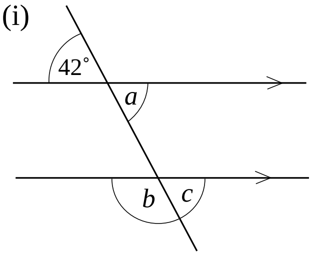
(g)



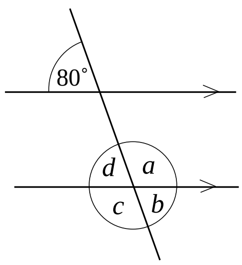
(h)



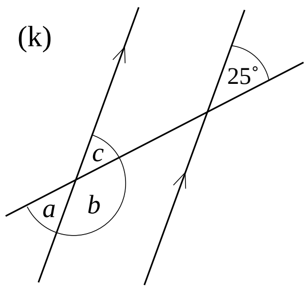
(i)



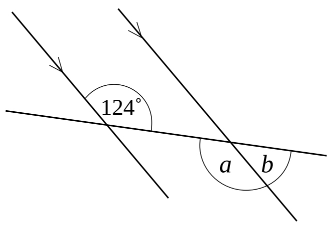
(j)



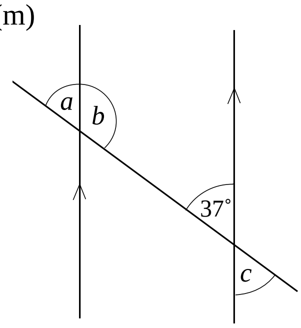
(k)



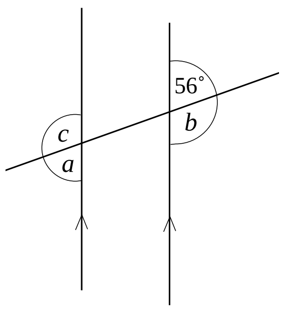
(l)



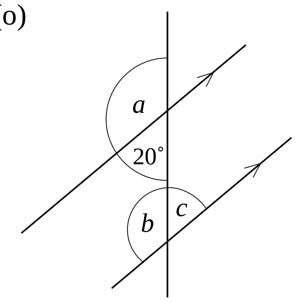
(m)



(n)

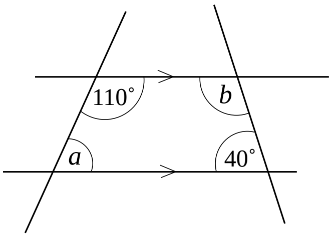


(o)

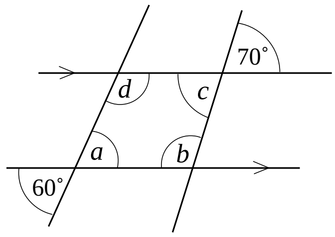


2. Find the size of the angles marked *a*, *b*, *c*, etc. in each of the diagrams below.

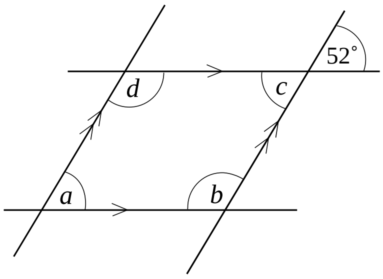
(a)



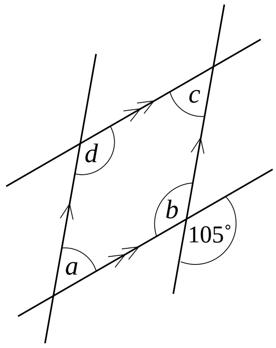
(b)



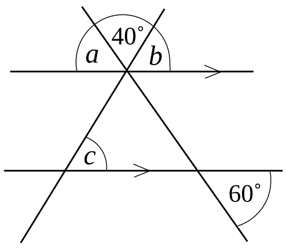
(c)



(d)



(e)



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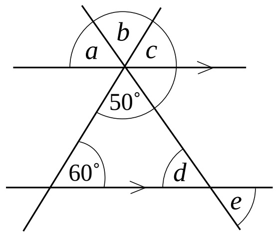
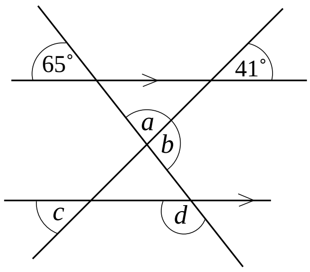
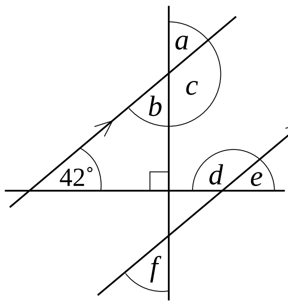


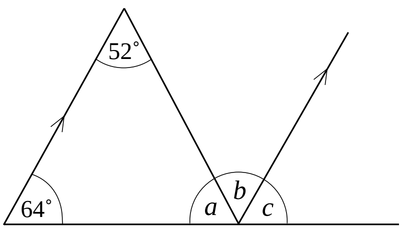
(g)



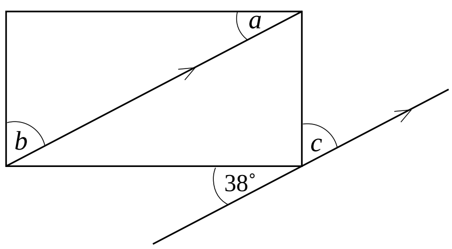
(h)



(i)

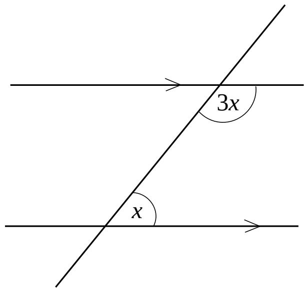


(j)

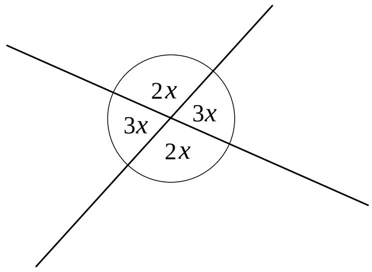


3. By considering each diagram, write down an equation and find the value of *x*.

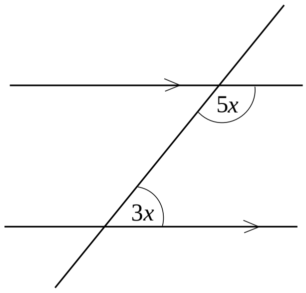
(a)



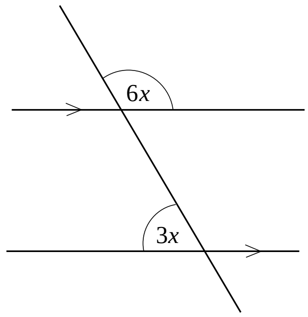
(b)



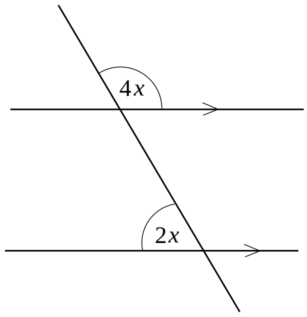
(c)



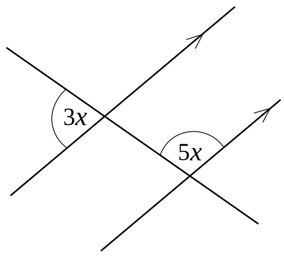
(d)



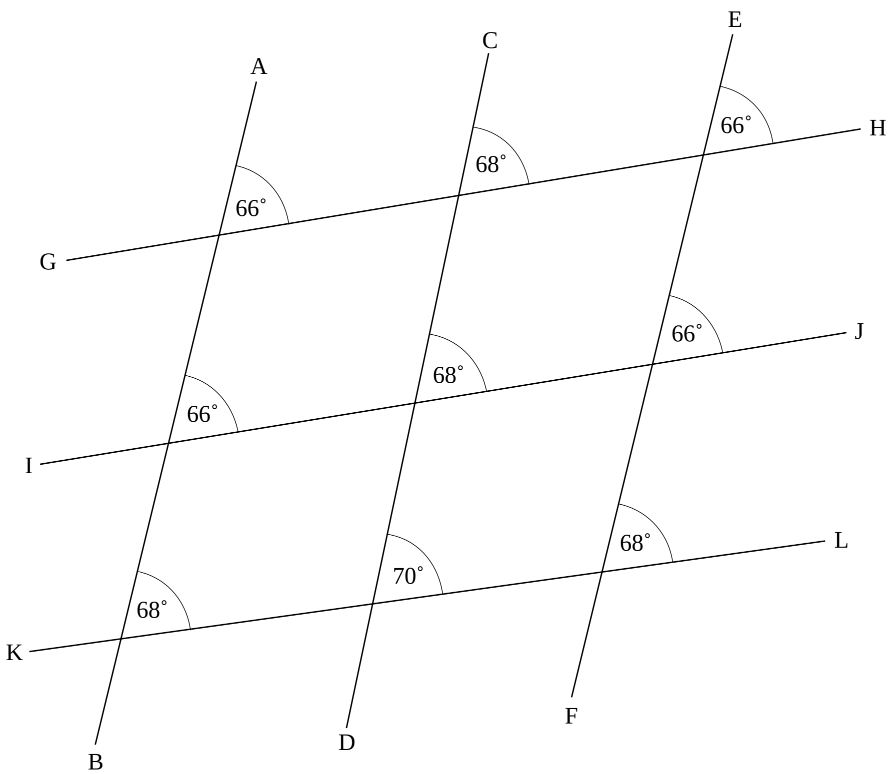
(e)



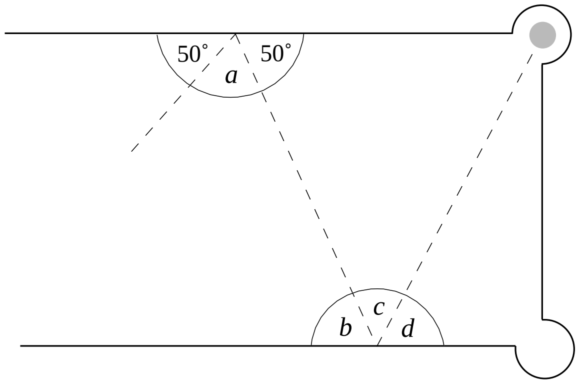
(f)



4. Which of the lines shown below are parallel?



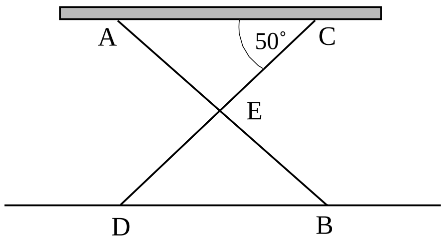
5. The diagram shows the path of a pool ball as it bounces off cushions on opposite sides of a pool table.



(a) Find the angles *a* and *b*.

(b) If, after the second bounce, the path is parallel to the path before the first bounce, find *c* and *d*.

6. A workbench is standing on a horizontal floor. The side of the workbench is shown.



The legs AB and CD are equal in length and joined at E. AE = EC

(a) Which two lines are parallel?

Angle ACD is $50°$.

(b) Work out the size of angle BAC giving a reason for your answer.

(SEG)

7. Here are the names of some quadrilaterals.

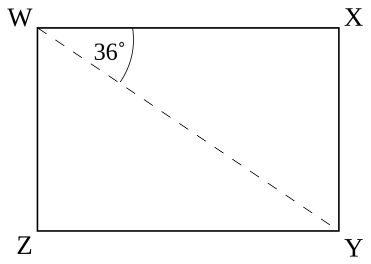
Square
Rectangle
Rhombus
Parallelogram
Trapezium
Kite

(a) Write down the names of the quadrilaterals which have two pairs of parallel sides.

(b) Write down the names of the quadrilaterals which must have two pairs of equal sides.

(LON)

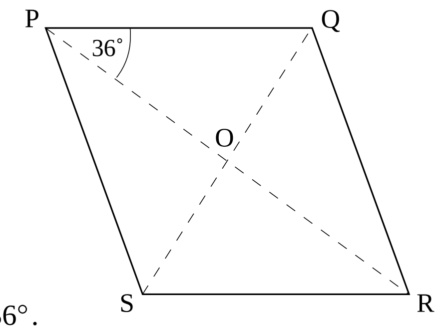
8. WXYZ is a rectangle.



Not to scale

(a) Angle XWY = 36°.
Work out the size of angle WYZ, giving a reason for your answer.

PQRS is a rhombus.



Not to scale

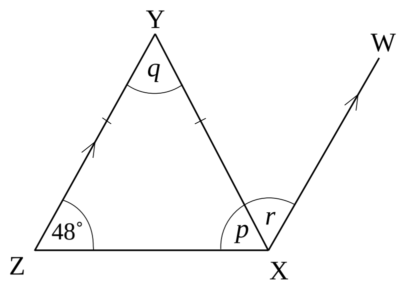
(b) Angle QPR = 36°.

The diagonals PR and QS cross at O.

Work out the size of angle PQS, giving a reason for your answer.

(SEG)

9. In the diagram, XY = ZY and ZY is parallel to XW.



Not to scale

(a) Write down the size of angle p.

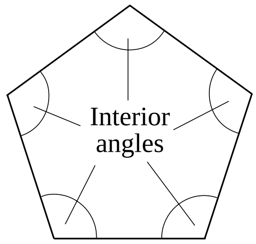
(b) Calculate the size of angle q. Give a reason for your answer.

(c) Give a reason why angle q = angle r.

(SEG)

3.5 Angle Symmetry in Polygons

Regular polygons will have both line and rotational symmetry. This symmetry can be used to find the *interior* angles of a regular polygon.

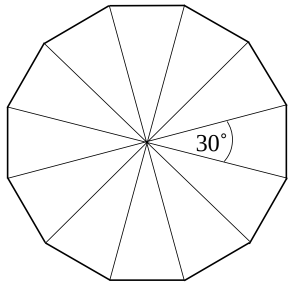


Worked Example 1

Find the interior angle of a regular dodecagon.

Solution

The diagram shows how a regular dodecagon can be split into 12 isosceles triangles.



As there are $360°$ around the centre of the dodecagon, the centre angle in each triangle is

$$\frac{360°}{12} = 30° \cdot$$

So the other angles of each triangle will together be

$$180° - 30° = 150° .$$

Therefore each of the other angles will be

$$\frac{150°}{2} = 75° .$$

As two adjacent angles are required to form each interior angle of the dodecagon, each interior angle will be

$$75° \times 2 = 150° .$$

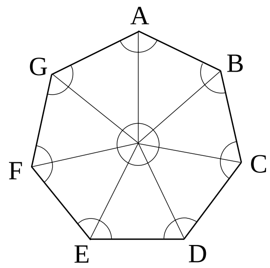
As there are 12 interior angles, the sum of these angles will be $12 \times 150° = 1800°$.

Worked Example 2

Find the sum of the interior angles of a heptagon.

Solution

Split the heptagon into 7 isosceles triangles.



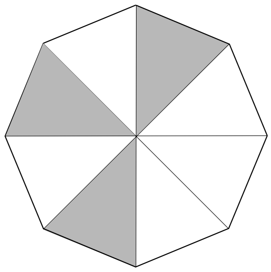
Each triangle contains three angles which add up to $180°$, so the total of all the marked angles will be

$$7 \times 180° = 1260° .$$

However the angles at the point where all the triangles meet should not be included, so the sum of the interior angles is given by

$$1260° - 360° = 900° .$$

Worked Example 3

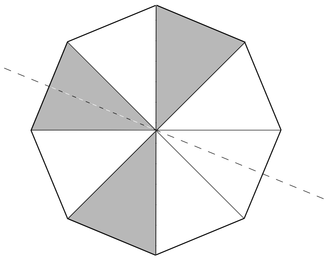


(a) Copy the octagon shown in the diagram and draw in any lines of symmetry.

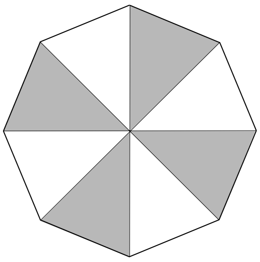
(b) Copy the octagon and shade in extra triangles so that it now has rotational symmetry.

Solution

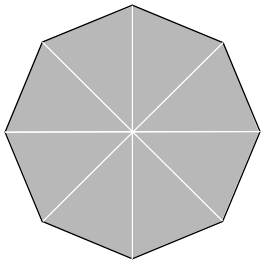
(a) There is only one line of symmetry as shown in the diagram.



(b) The original octagon has no rotational symmetry.



By shading the extra triangle shown, it has rotational symmetry of order 4.



By shading all the triangles, it has rotational symmetry of order 8.

Exercises

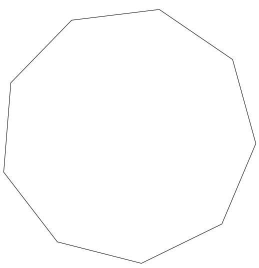
1. Find the interior angle for a regular:

   (a) pentagon (b) hexagon

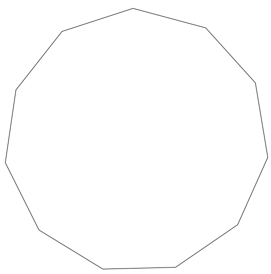
   (c) octagon (d) decagon (10 sides).

2. Find the sum of the interior angles in each polygon shown below.

   (a)



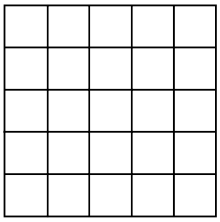
   (b)

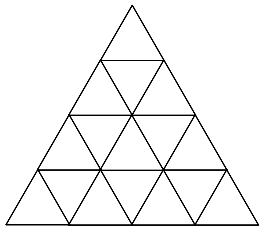


3. Which regular polygons have interior angles of:

(a)	90°	(b)	120°	(c)	108°
(d)	140°	(e)	60°	(f)	144° ?

4. Make 3 copies of each shape below.





and shade parts of them, so that:

(a) they have line symmetry, but no rotational symmetry;

(b) they have line symmetry and rotational symmetry;

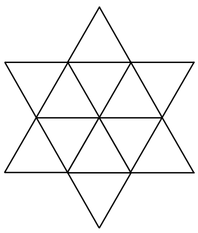
(c) they have rotational symmetry, but no line symmetry.

In each case draw in the lines of symmetry and state the order of rotational symmetry.

5. (a) Draw a shape that has rotational symmetry of order 3 but no line symmetry.

(b) Draw a shape that has rotational symmetry of order 5 but no line symmetry.

6. (a)



For this shape, is it possible to shade smaller triangle so that is has rotational symmetry of order:

(i) 2 (ii) 3 (iii) 4

with no lines of symmetry?

(b) Is it possible to shade smaller triangles so that the shape has

(i) 1 (ii) 2 (iii) 3

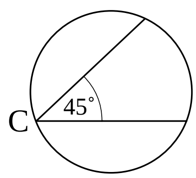
lines of symmetry and no rotational symmetry?

Investigation

The diagram shows the plan of a circular hall for a jewellery exhibition. C is a hidden video camera which scans an angle of 45˚.

How many video cameras must be installed on the walls of the hall so that they will cover the whole hall completely?

Indicate the position where each video camera must be mounted.



How many video cameras do you need if each can scan

(a) 35˚ (b) 60˚ (c) 90˚ (d) 100˚?

7. (a) A polygon has 9 sides. What is the sum of the interior angles?

(b) Copy and complete the table below.

Shape	**Sum of interior angles**
Triangle	180°
Square	
Pentagon	
Hexagon	720°
Heptagon	
Octagon	

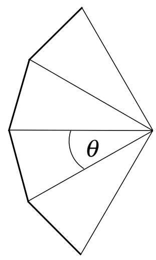
(c) Describe a rule that could be used to calculate the sum of the interior angles for a polygon with n sides.

(d) Find the sum of the interior angles for a 14-sided polygon.

(e) The sum of the interior angles of a polygon is $1260°$. How many sides does the polygon have?

8. (a) A regular polygon with n sides is split into isosceles triangles as shown in the diagram.

Find a formula for the size of the angle marked θ.



(b) Use your answer to part (a) to find a formula for the interior angle of a regular polygon with n sides.

(c) Use your formula to find the interior angle of a polygon with 20 sides.

9.

(a) Write down the order of rotational symmetry of this rectangle.

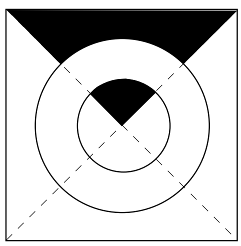
(b) Draw a shape which has rotational symmetry of order 3.

(c) (i) How many lines of symmetry has a regular pentagon?

(ii) What is the size of one exterior angle of a regular pentagon?

(NEAB)

10.

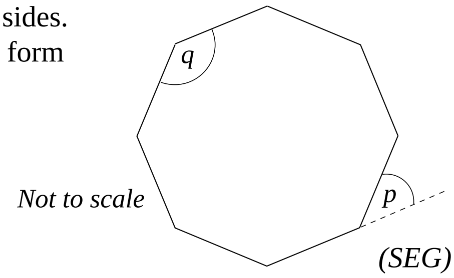


The picture shows a large tile with only part of its pattern filled in.

Complete the picture so that the tile has 2 lines of symmetry and rotational symmetry of order 2.

(NEAB)

11. A regular octagon, drawn opposite, has eight sides. One side of the octagon has been extended to form angle p.

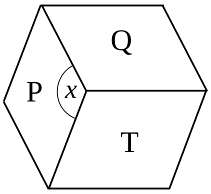


(a) Work out the size of angle p.

(b) Work out the size of angle q.

(SEG)

12.



The diagram shows three identical rhombuses, P, Q and T.

(a) Explain why angle x is $120°$.

(b) Rhombus Q can be rotated onto rhombus T.

(i) Mark a centre of rotation.

(ii) State the angle of rotation.

(c) Write down the order of rotational symmetry of

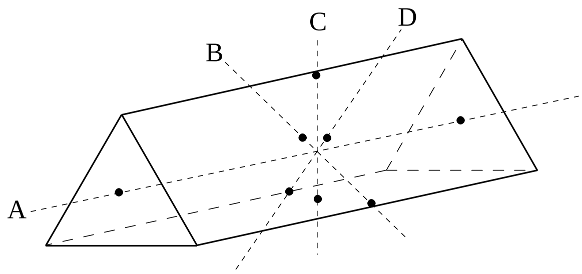
(i) a rhombus (ii) a regular hexagon.

(d) The given shape could also represent a three dimensional shape. What is this shape?

(NEAB)

3.6 Symmetry Properties of 3D Shapes

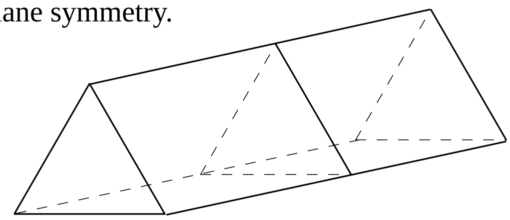
When describing the symmetry of three dimensional objects, it is possible to have rotational symmetry about an axis. The triangular prism shown below has rotational symmetry about the 4 axes shown.



There is symmetry of order 3 about the axis labelled A.

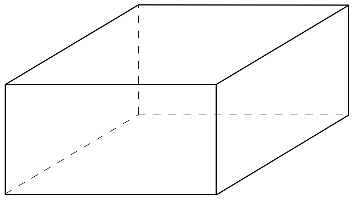
There is symmetry of order 2 about the other axes labelled B, C and D.

The prism also has plane symmetry.



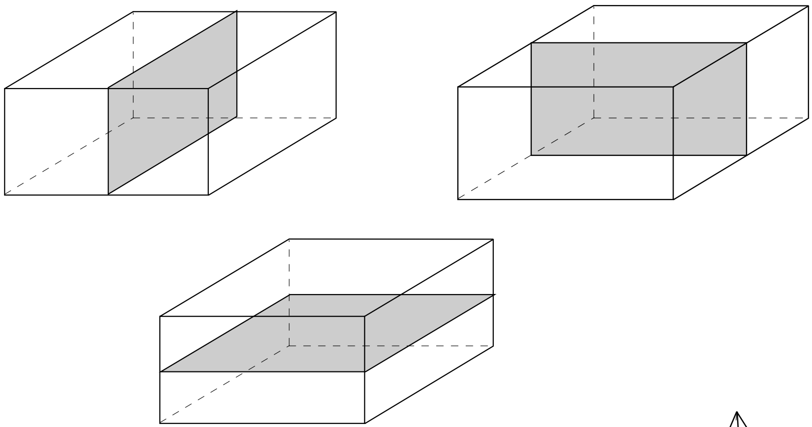
Worked Example 1

Draw diagrams to show the positions of the planes of symmetry of the rectangular cuboid shown in the diagram.



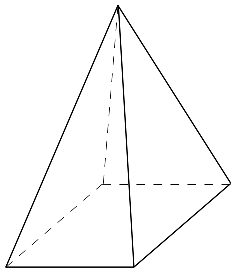
Solution

There are three planes of symmetry. Two planes are vertical and the other is horizontal as shown below.



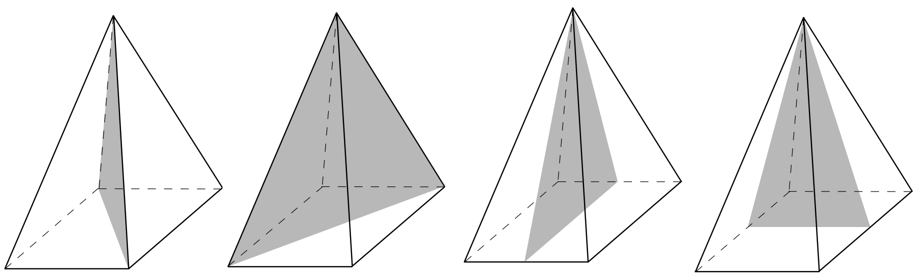
Worked Example 2

Using diagrams where appropriate, describe all the symmetries of the square-based pyramid shown in the diagram.



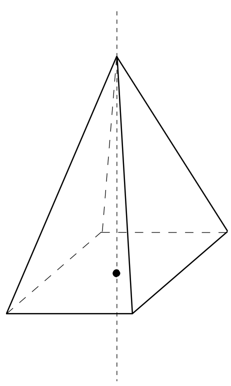
Solution

There are four vertical planes of symmetry as shown in the diagrams below.



There is also one axis of rotational symmetry.

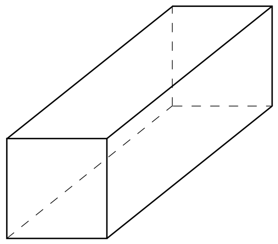
This is a vertical line through the centre of the base.
The order of the symmetry about this line is 4.



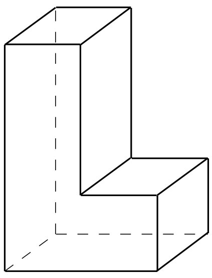
Exercises

1. For each solid below draw diagrams to show all the planes of symmetry.

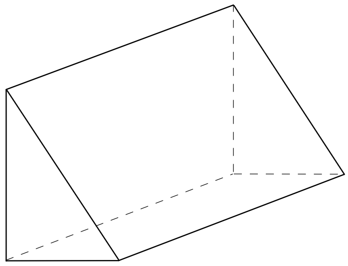
(a)



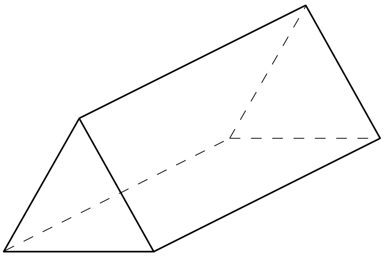
(b)



(c)

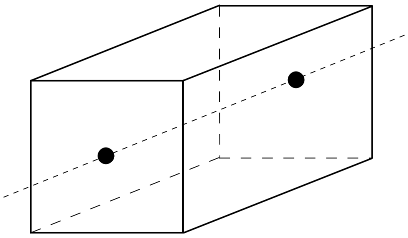


(d)

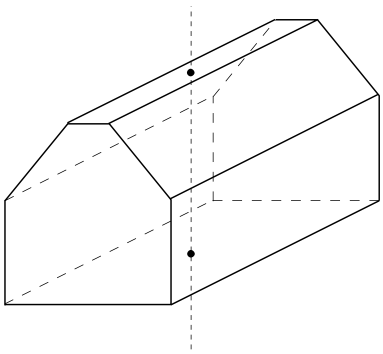


2. State the order of rotational symmetry about the axis shown in each diagram.

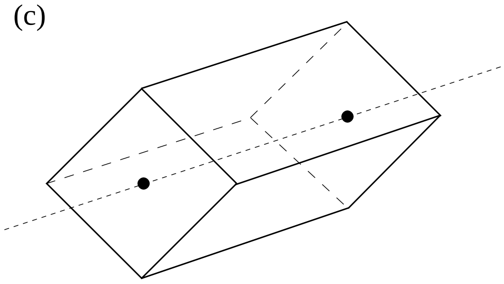
(a)



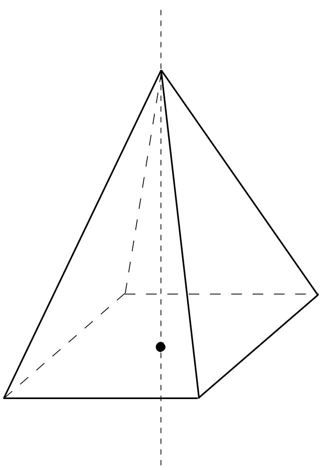
(b)



(c)

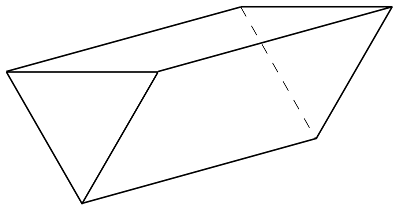


(d)

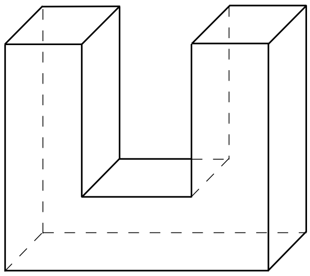


3. If possible, draw an axis of symmetry for each solid below, so that there is rotational symmetry of order 2 about the axis.

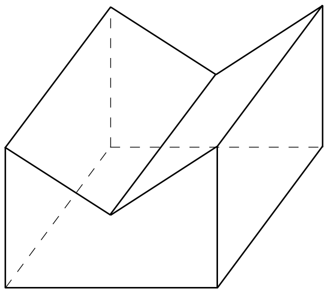
(a)



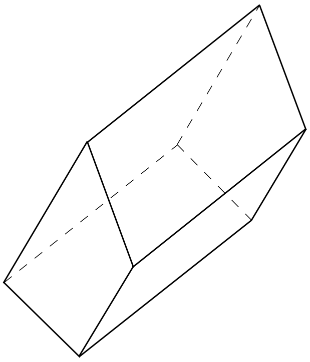
(b)



(c)

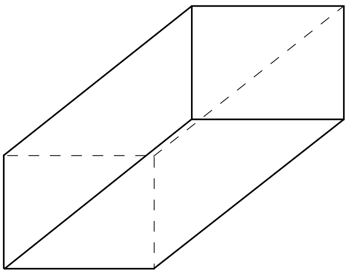


(d)

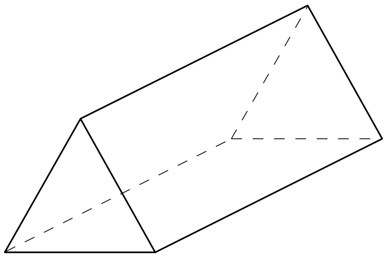


4. For each solid below, show or describe all the axes of symmetry.
Also state the order of the rotational symmetry about each line.

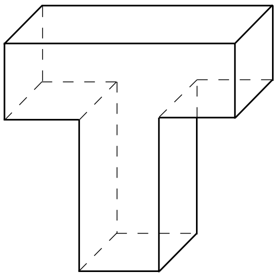
(a)



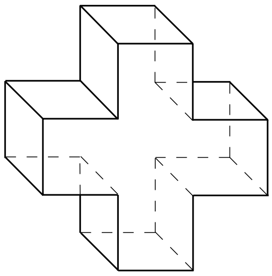
(b)



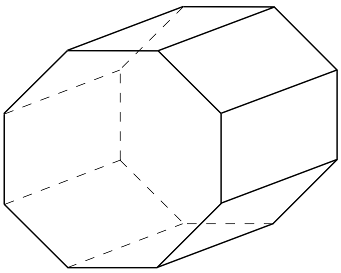
(c)



(d)



5. Describe all the symmetries of this shape.



6. Describe the symmetries of a cuboid that has two square faces.

7. For each shape below describe fully how many planes of symmetry there are, the number of axes of symmetry and the order of the rotational symmetry:

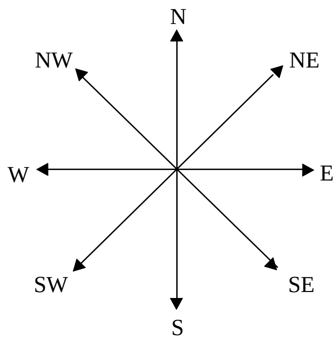
   (a) cylinder (b) sphere
   (c) hemisphere (d) cone

8. Draw objects which have:

   (a) one plane of symmetry
   (b) two planes of symmetry
   (c) one axis of rotational symmetry
   (d) two axes of rotational symmetry.

3.7 Compass Bearings

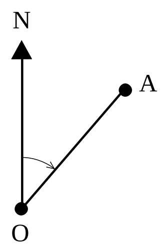
When describing a direction, the points of a compass can be useful, e.g. S or SW.



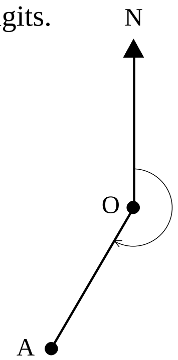
A *bearing* can also be used, especially in navigation and by people walking on rough or open moorland or hills.

Note

Bearings are always measured clockwise from North and use 3 digits.



The bearing of A from O is $050°$.



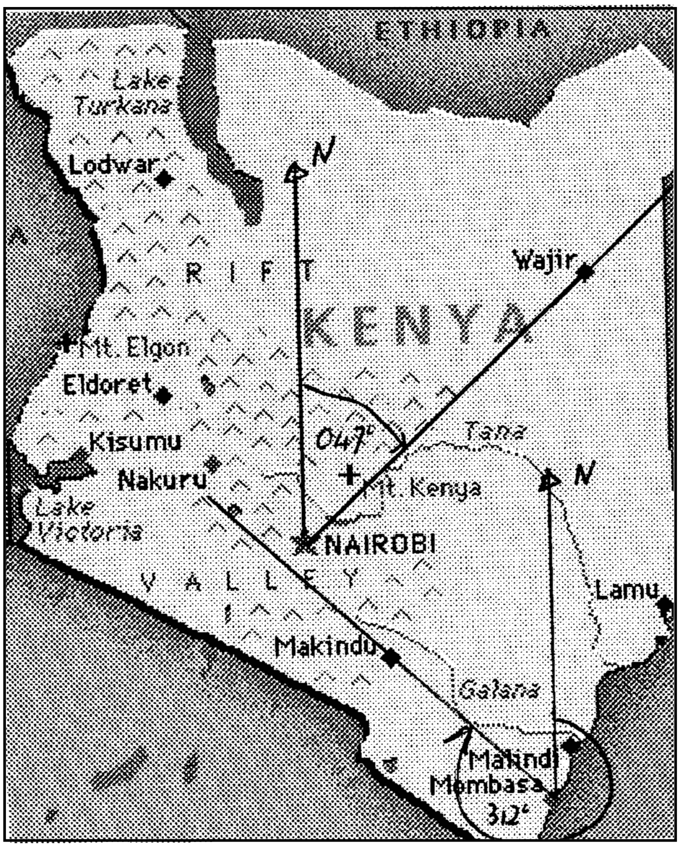
The bearing of A from O is $210°$.

Worked Example 1

On a map of Kenya, find the bearings of

(a) Wajir from Nairobi
(b) Makindu from Mombasa.

Solution



Map of Kenya

(a) First draw in a North line at Nairobi and another line from Nairobi to Wajir.

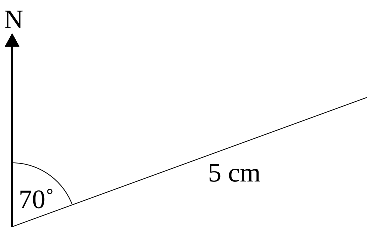
Then measure the angle clockwise from North to the second line. In this case the angle is $47°$ so the bearing is $047°$.

(b) Draw a North line at Mombassa and a line from Mombassa to Makindu.
The bearing can then be measured as $312°$.

Worked Example 2

A boat sails for 500 miles on a bearing of $070°$ and then sails a further 700 miles on a bearing of $200°$. Find the distance of the boat from its starting point and the bearing that would have taken it straight there.

Solution

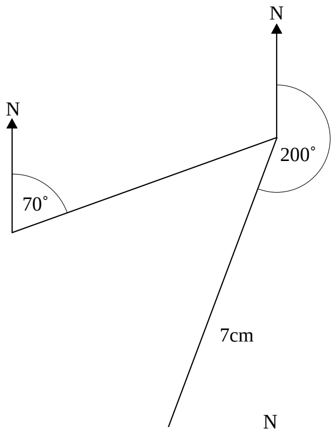


To find the solution use a scale drawing.

1. Draw a north arrow at the starting point.
2. Measure an angle of $70°$ from North.
3. Draw a line 5 cm long. (1 cm represents 100 m)

4. Draw a second North arrow and measure an angle of $200°$.

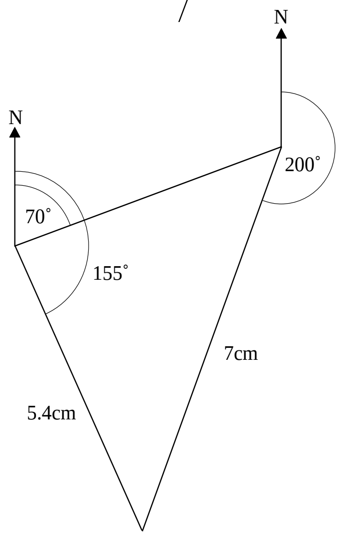
5. Draw a line 7 cm long.



6. Join the final point to the starting point and measure the distance.

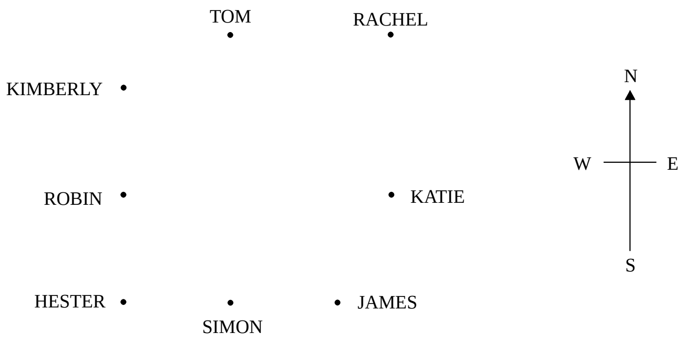
It is 5.4 cm, which represents 540 m.

7. The bearing can also be measured as $155°$.



Exercises

1. The diagram shows the positions of 8 children.



(a) Who is directly south of Rachel?

(b) If Kimberley walks SE, whom will she meet?

(c) If Rachel walks SW, whom will she meet?

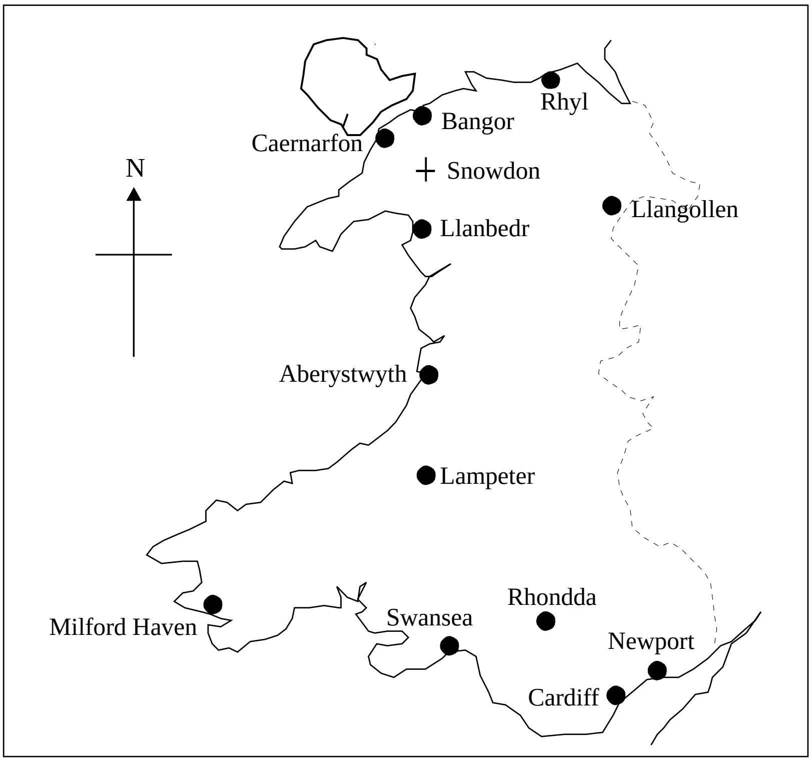
(d) Who is directly west of Katie?

(e) Who is NW of Katie?

(f) Who will Simon meet if he walks NW?

(g) In what direction should Hester walk to find Rachel?

2. The map shows some towns and cities in Wales.

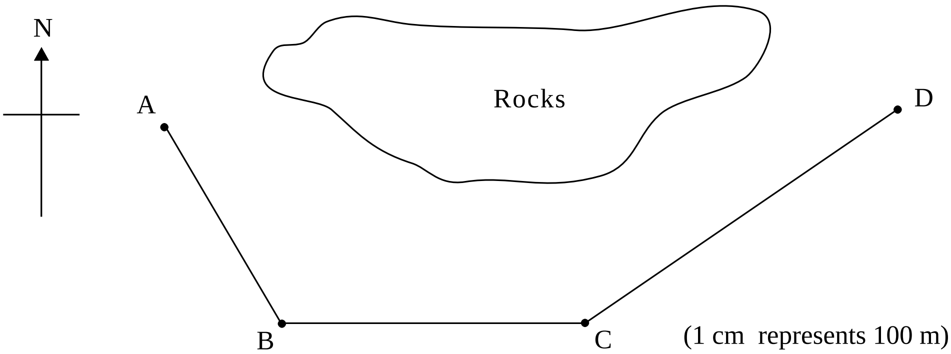


Write down the bearing of each of the following places from Mount Snowdon.

(a) Llangollen (b) Newport (c) Swansea

(d) Bangor (e) Milford Haven (f) Aberystwyth

3. In order to avoid an area of dangerous rocks a yacht sails as shown in the diagram.

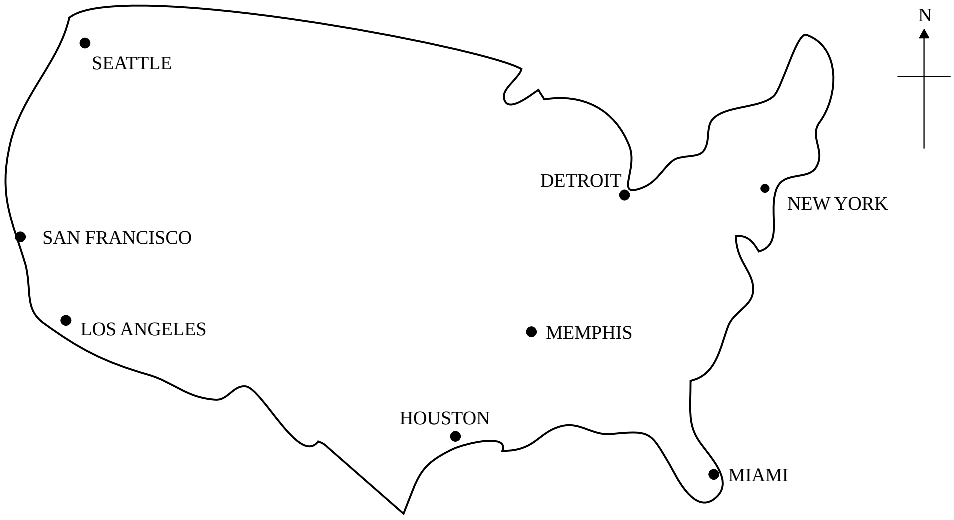


(a) Find the bearing of the yacht as it sails from

(i) A to B (ii) B to C (iii) C to D.

(b) How much further does the yacht travel to avoid the rocks?

4. A rough map of the USA is shown below.



Find the bearings of:

(a) Miami from Memphis.

(b) Detroit from New York

(c) Los Angeles from Detroit.

(d) Seattle from Houston.

(e) Memphis from San Francisco.

5. Use a scale drawing to find the answers to each of the following problems.

(a) If a man walks 700 m on a bearing of $040°$, how far north and how far east is he from his starting point?

(b) A dog runs 50 m on a bearing of $230°$ and then runs north until he is west of his starting point. How far is the dog from its starting point?

(c) A helicopter flies 80 km north and then 20 km SW. What bearing would have taken the helicopter directly to its final position?
How far is the helicopter from its starting point?

(d) A boat travels 500 m NE and then 500 m south. What bearing would take the boat back to its starting point?

(e) A plane flies 300 km west and then a further 200km SW. It then returns directly to its starting point.
On what bearing should it fly and how far does it have to travel?

6. Use a scale drawing to illustrate each of the following journeys.
Describe the return journey in each case using a single bearing and distance.

(a) 120 m on $090°$ followed by 120 m on $180°$.

(b) 500 km on $045°$ followed by 200 km on $270°$.

(c) 300 km on $220°$ followed by 300 km on $170°$.

(d) 25 km on $330°$ followed by 30 km on 170.

(e) 10 km on $160°$ followed by 2 km on $300°$.

(f) 30 km on $120°$ followed by 30 km on $270°$.

(g) 1000 m on $050°$ followed by 1200 m on $310°$.

7. A ship sails from a point A to another point B, 8000 m due east of A.
It then sails in another direction and arrives at a point C, 10 000 m SE of A.

On what bearing did the ship sail on the second stage of the journey and how far did it travel?

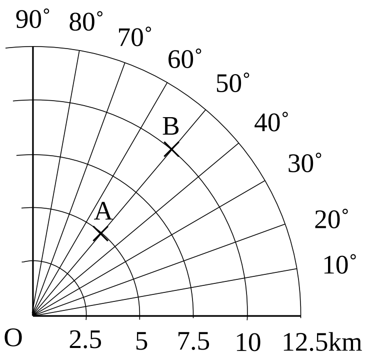
8. Here is a map.

(a) Name the town north of Manchester.

(b) Name the town south west of Birmingham.

(LON)

9. The position of ship A from O is (5 km, 50°).



What is the position of ship B from A?

(SEG)

Investigation

Draw a rectangle of any size. Use your ruler to locate the mid-points of the sides. Join these mid-points to form a new quadrilateral.

What is the name of the quadrilateral you have obtained?

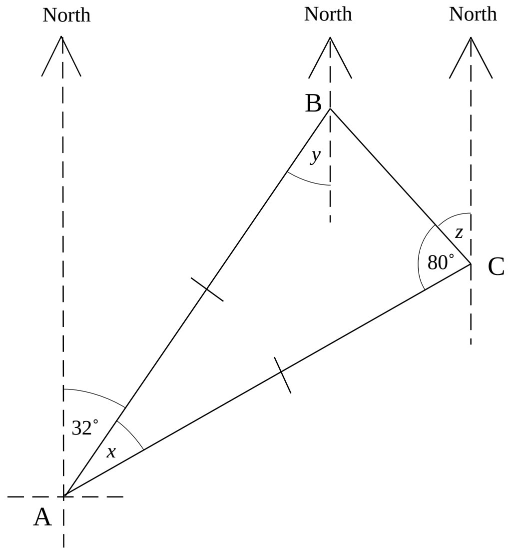
Repeat the above by drawing

(a) a trapezium (b) a parallelogram (c) a kite

(d) a rhombus (e) a quadrilateral of 4 unequal lengths.

What conclusion can you draw from these?

10.



The diagram is not drawn to scale.

The diagram shows the position of three places, A, B and C.
AB is the same length as AC.

(a) (i) Calculate the size of the angle marked x.

(ii) Explain why the angle marked y is equal to $32°$.

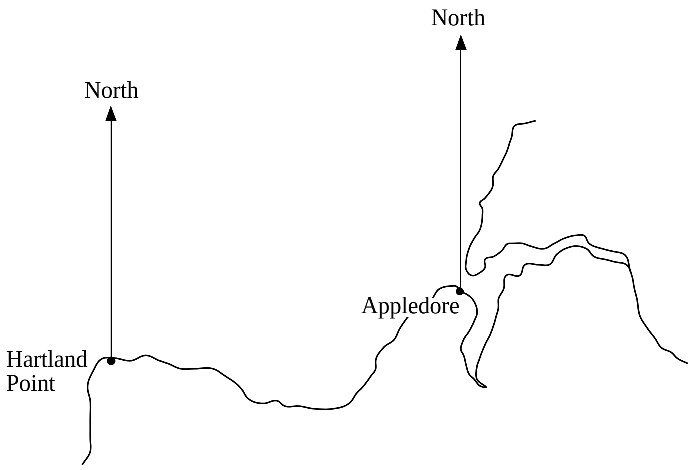
(iii) Calculate the size of the angle marked z.

(b) Use your answers to (a) to calculate the bearing of

(i) C from A (ii) A from B (iii) B from C.

(NEAB)

11. The diagram below shows a map of part of the North Devon coast.



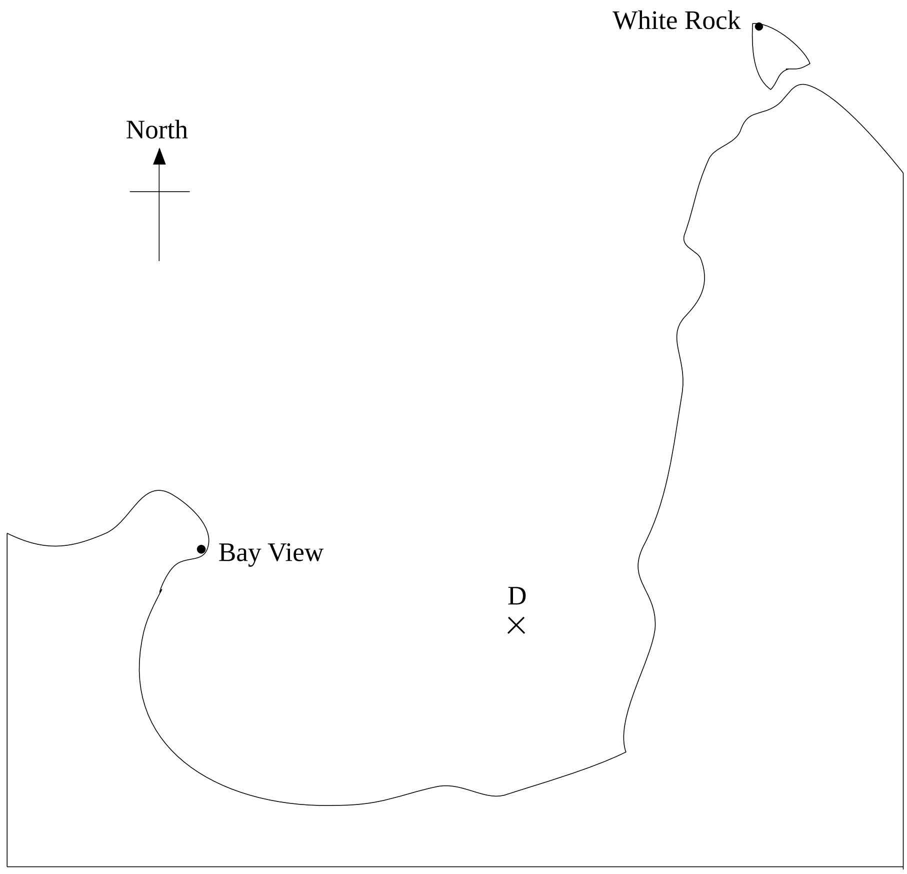
The bearing of a ship from Hartland Point is $070°$.

Its bearing from Appledore is $320°$.

Trace the diagram. Showing your construction lines, mark on your tracing the position of the ship and label its position with the letter S.

(MEG)

12. The diagram shows a bay in which yachts are moored.
    The diagram has been drawn to a scale of 5 cm to 1 km.



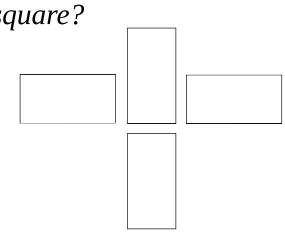
(a) The yacht *Daresa* is moored at D.
    Measure the bearing of this yacht from Bay View.

(b) The yacht *Wet-n-Windy* is moored 1.2 km from White Rock on a bearing of $210°$. Trace the diagram and mark with a cross the position of this yacht on the diagram.

(SEG)

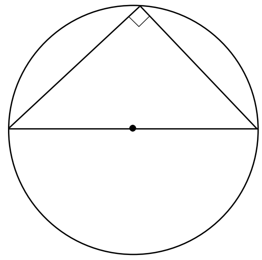
Just for fun

Four rectangular cards of identical size are arranged as shown below. Can you move only one card so as to form a square?



3.8 Angles and Circles 1

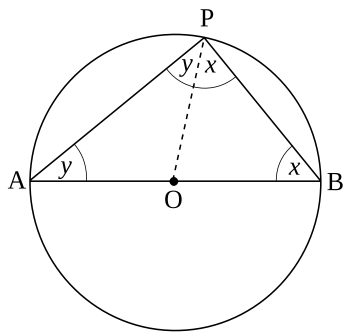
The following results are true in any circle.



When a triangle is drawn in a semi-circle as shown;
the angle on the perimeter is always a right angle.

Proof

Join the centre, O, to the point, P, on the perimeter.



Since

$$OB = OP \qquad \text{(equal radii)}$$

then

$$\text{angle } OBP = \text{angle } OPB \qquad (= x, \text{ say})$$

Similarly, triangle AOP is also isosceles and

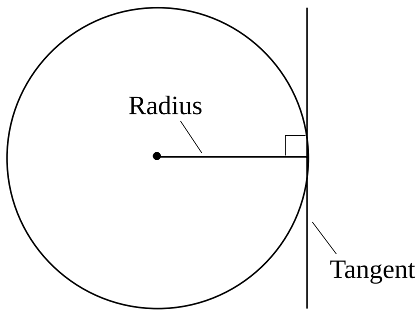
$$\text{angle } OAP = \text{angle } APO \qquad (= y, \text{ say}).$$

In triangle ABP, the sum of the angles must be $180°$.
Then

$$\begin{aligned} y + x + (x + y) &= 180° \\ 2x + 2y &= 180° \qquad \text{(collecting like terms)} \\ x + y &= 90°. \qquad (\div 2) \end{aligned}$$

But angle APB $= x + y$, and this is a right angle.

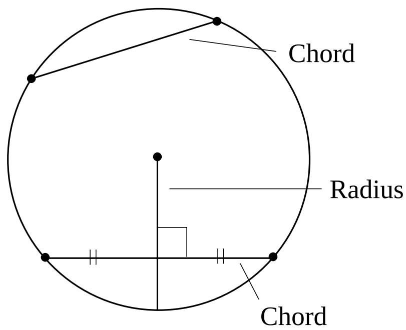
A *tangent* is a line that just touches a circle.
It is always perpendicular to the radius.

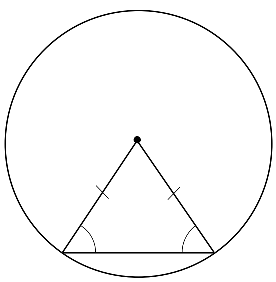


A *chord* is a line joining any two points on the circle.

The *perpendicular bisector* is a second line that cuts the first line in half and is at right angles to it.

The perpendicular bisector of a chord is always a radius of the circle.

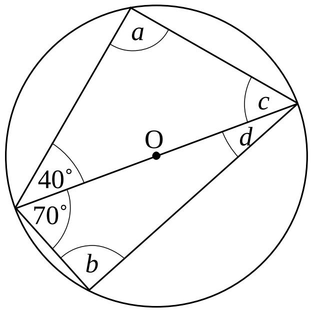




When the ends of a chord are joined to the centre of a circle, an isosceles triangle is formed, so the two base angles marked are equal.

Worked Example 1

Find the angle marked with letters in the diagram, if O is the centre of the circle.



Solution

As both triangles are in semi-circles, angles *a* and *b* must each be $90°$.

The other angles can be found because the sum of the angles in each triangle is $180°$.

For the top triangle,

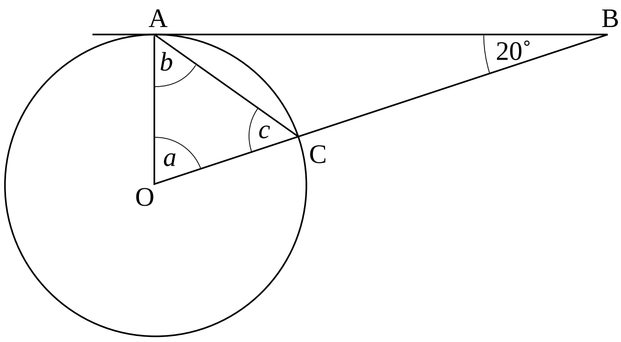
$$\begin{aligned} 40° + 90° + c &= 180° \\ c &= 180° - 130° \\ &= 50°. \end{aligned}$$

For the bottom triangle,

$$\begin{aligned} 70° + 90° + d &= 180° \\ d &= 180° - 160° \\ &= 20°. \end{aligned}$$

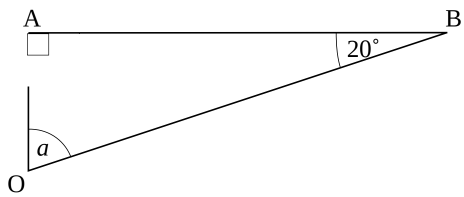
Worked Example 2

Find the angles a, b and c, if AB is a tangent and O is the centre of the circle.



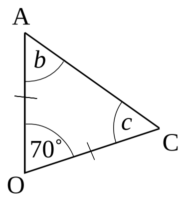
Solution

First consider the triangle OAB. As OA is a radius and AB is a tangent, the angle between them is $90°$. So



$$\begin{aligned} 90° + 20° + a &= 180° \\ a &= 180° - 110° \\ &= 70°. \end{aligned}$$

Then consider the triangle OAC. As OA and OC are both radii of the circle, it is an isosceles triangle with $b = c$.

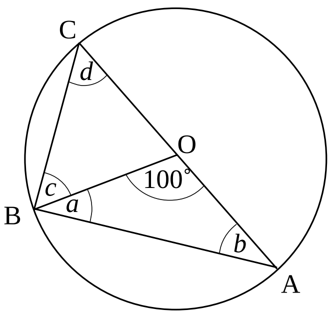


So
$$\begin{aligned} 2b + 70° &= 180° \\ 2b &= 110° \\ b &= 55° \end{aligned}$$

and $c = 55°$.

Worked Example 3

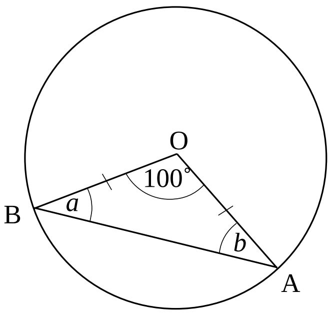
Find the angles marked in the diagram, where O is the centre of the circle.



Solution

First consider the triangle OAB.

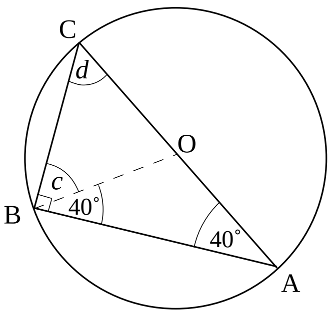
As the sides OA and OB are both radii, the triangle must be isosceles with $a = b$.



So $$a + b + 100° = 180°$$
but as $a = b$,
$$2a + 100° = 180°$$
$$2a = 80°$$
$$a = 40°$$
and $b = 40°$.

Now consider the triangle ABC.

As the line AC is a diameter of the circle, the angle ABC must be $90°$.
So
$$a + c = 90°$$
or
$$40° + c = 90°$$
$$c = 50°.$$



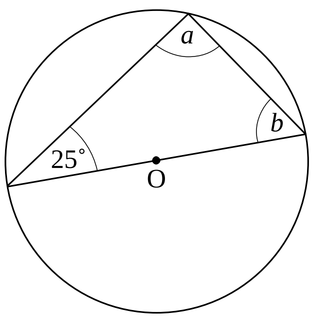
The angles in the triangle ABC must total $180°$, so

$$40° + 90° + d = 180°$$
$$d = 50°.$$

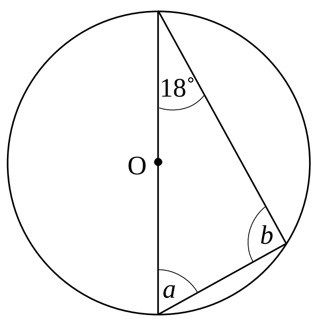
Exercises

1. Find the angles marked with a letter in each of the following diagrams. In each case the centre of the circle is marked O.

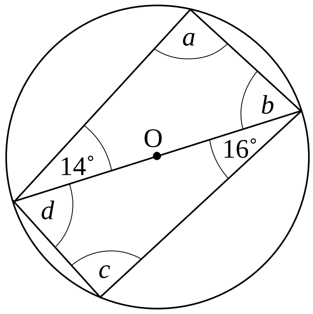
(a)



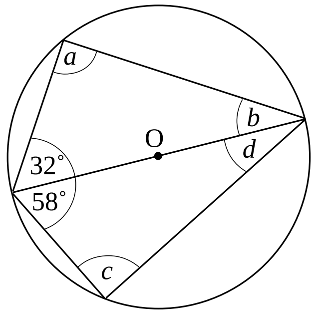
(b)



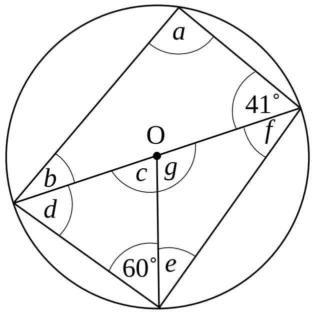
(c)



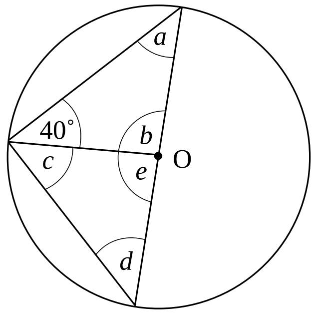
(d)



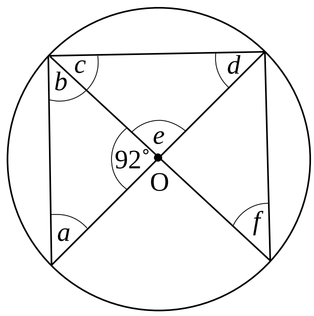
(e)



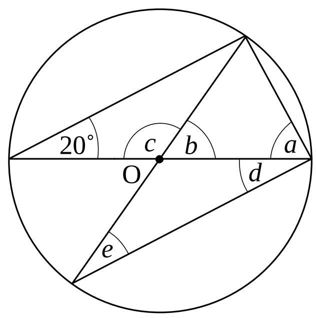
(f)



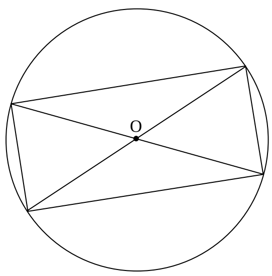
(g)



(h)

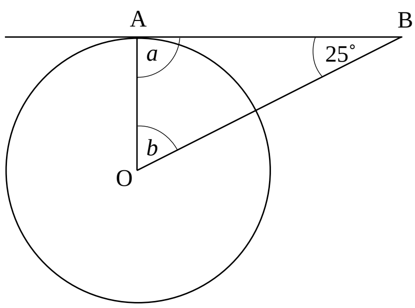


2. Copy the diagram below, and mark every right angle, if O is the centre of the circle.



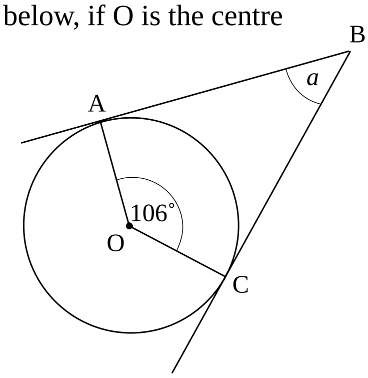
3. Find the angles marked with letters in each diagram below, if O is the centre of the circle.

(a)



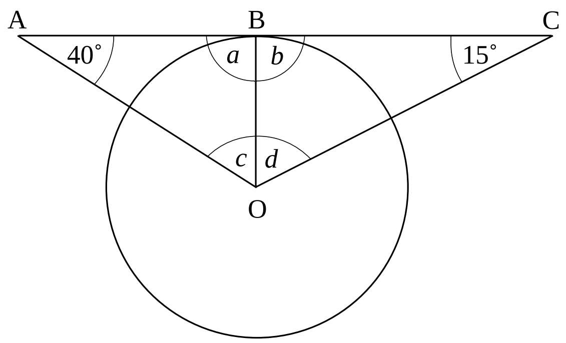
AB is a tangent

(b)



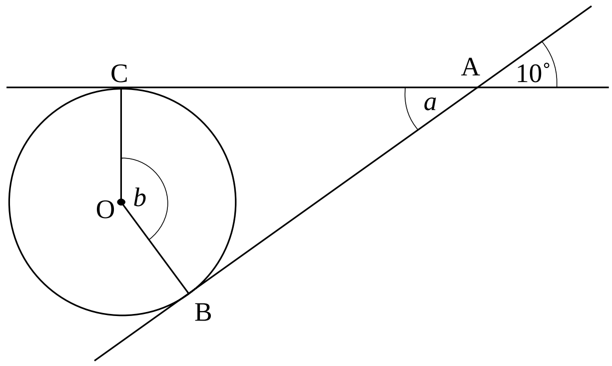
AB and BC are tangents

(c)



AC is a tangent

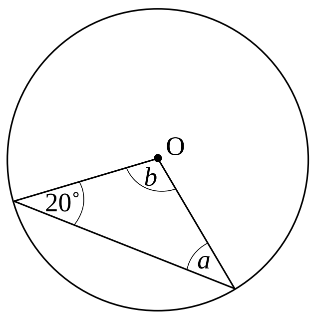
(d)



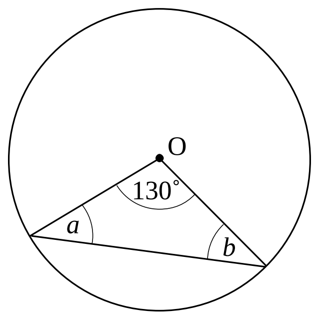
AB and AC are tangents

4. Find the angles marked with letters in each of the following diagrams, if O is the centre of the circle.

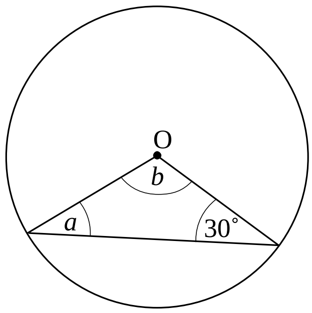
(a)



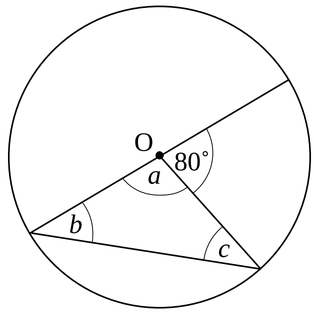
(b)



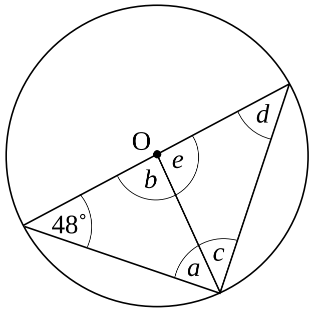
(c)



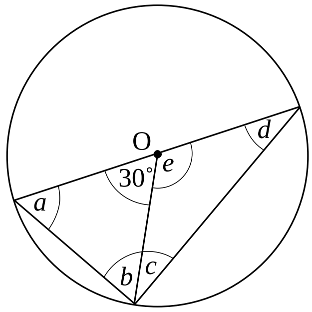
(d)



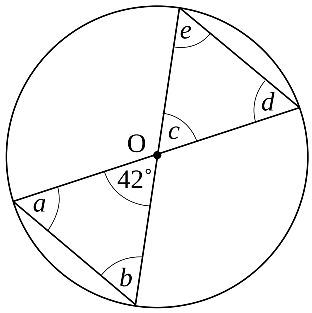
(e)



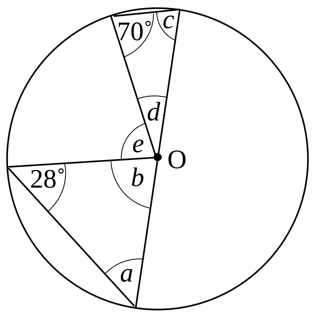
(f)



(g)

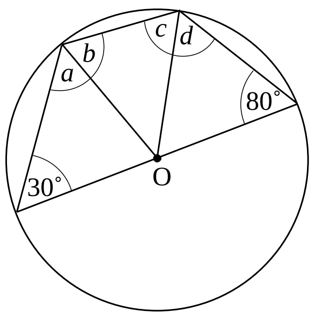


(h)

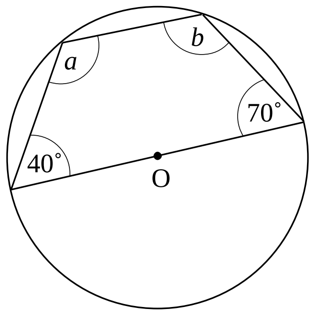


5. Find each of the marked angles if O is the centre of the circle.

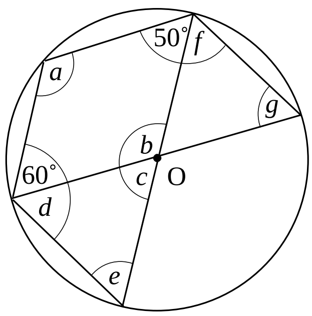
(a)



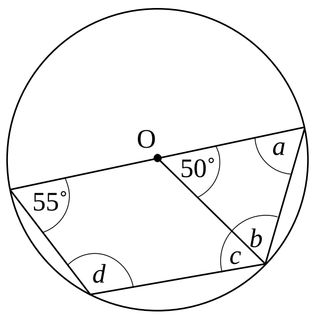
(b)



(c)

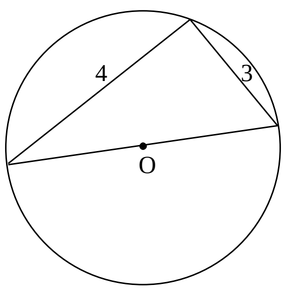


(d)

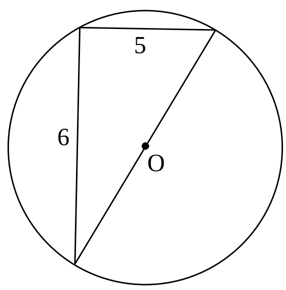


6. Find the diameter of each circle below, if O is the centre of the circle.

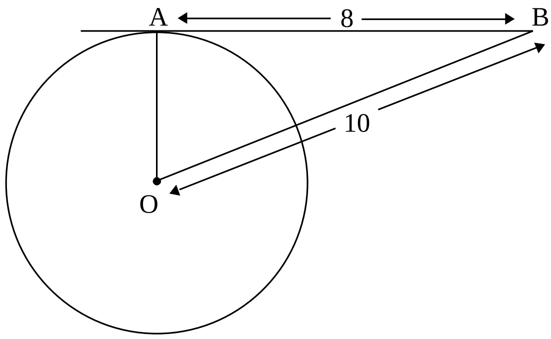
(a)



(b)

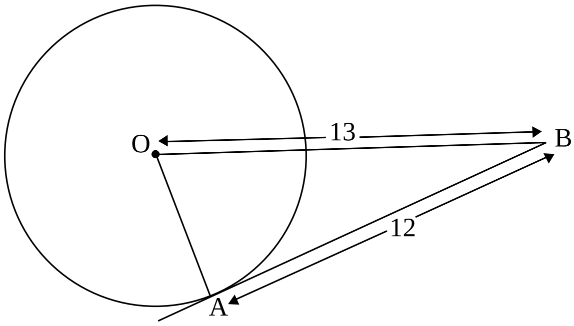


(c)



AB is a tangent

(d)

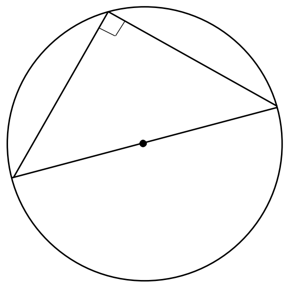


AB is a tangent

(SEG)

3.9 Angles and Circles 2

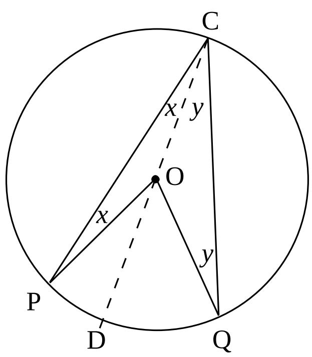
There are a number of important geometric results based on angles in circles. (The first you have met already.)



1. Any angle subtended on the circumference from a diameter is a right angle.

2. The angle subtended by an arc, PQ, at the centre is twice the angle subtended on the perimeter.

Proof



OP = OC (equal radii), so

angle CPO = angle PCO (= x, say).

Similarly,

angle CQO = angle QCO (= y, say).

Now, extending the line CO to D, say, note that

$$\text{angle POD} = x + x = 2x$$

and, similarly,

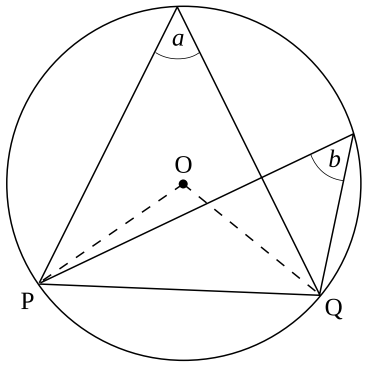
$$\text{angle QOD} = y + y = 2y.$$

Hence,

$$\begin{aligned} \text{angle POQ} &= 2x + 2y \\ &= 2(x+y) \\ &= 2 \times \text{angle PCQ} \end{aligned}$$

as required.

3. Angles subtended at the circumference by a chord (on the same side of the chord) are equal;
   that is, in the diagram $a = b$.



Proof

The angle at the centre is $2a$ or $2b$ (according to the first result).

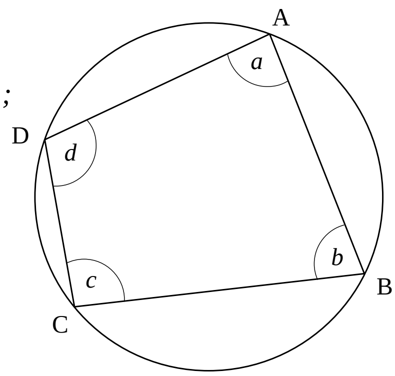
Thus $2a = 2b$ or $a = b$, as required.

4. In *cyclic quadrilaterals* (quadrilaterals where all 4 vertices lie on a circle), opposite angles sum to $180°$;
   that is

$$a + c = 180°$$

and

$$b + d = 180°.$$

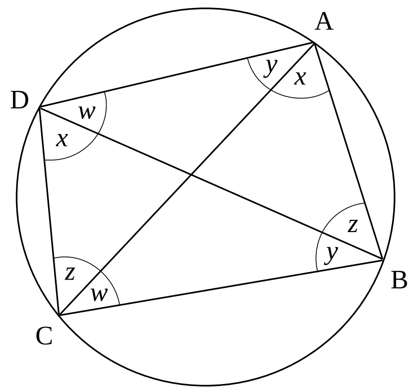


Proof

Construct the diagonals AC and BD, as below.

Then label the angles subtended by AB as w; that is

$$\text{angle ADB} = \text{angle ACB} \quad (= w).$$



Similarly for the other chords, the angles being marked x, y and z as shown.

Now, in triangle ABD, the sum of the angles is $180°$, so

$$w + z + (x + y) = 180°.$$

You can rearrange this as

$$(x + w) + (y + z) = 180°,$$

which shows that

$$\text{angle CDA} + \text{angle CBA} = 180°,$$

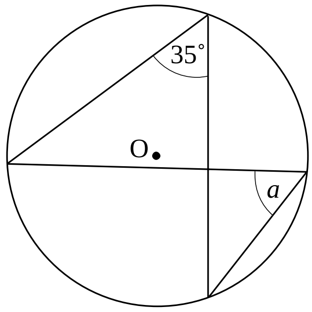
proving one of the results.

The other result follows in a similar way.

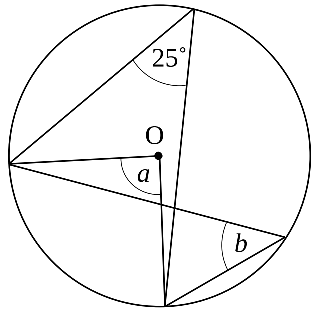
Worked Example 1

Find the angles marked in the diagrams. In each case O is the centre of the circle.

(a)



(b)



Solution

(a) As both angles are drawn on the same chord, the angles are equal, so

$$a = 35°.$$

(b) Angle b and the $25°$ angle are drawn on the same chord, so

$$b = 25°.$$

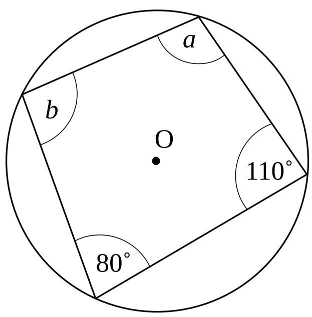
Angle a is drawn at the centre O on the same chord as the $25°$ angle, so

$$\begin{aligned} a &= 2 \times 25° \\ &= 50°. \end{aligned}$$

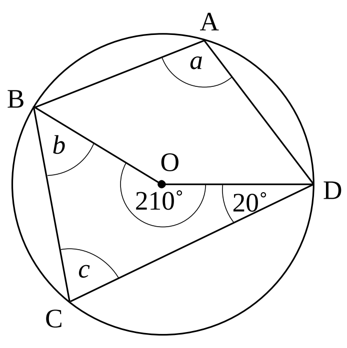
Worked Example 2

Find the angles marked in the diagrams. O is the centre of the circle.

(a)



(b)



Solution

(a) Opposite angles in a cyclic quadrilateral add up to $180°$. So

$$\begin{aligned} a + 80° &= 180° \\ a &= 100° \end{aligned}$$

and

$$\begin{aligned} b + 110° &= 180° \\ b &= 70°. \end{aligned}$$

(b) Consider the angles a and $210°$. Since the angle at the centre is double the angle in a segment drawn on the same arc,

$$\begin{aligned} 2a &= 210° \\ a &= 105°. \end{aligned}$$

Angles a and c add up to $180°$ because they are opposite angles in a cyclic quadrilateral.

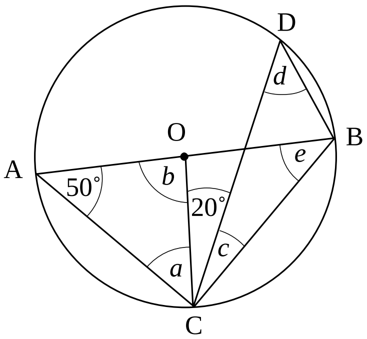
$$\begin{aligned} a + c &= 180° \\ 105° + c &= 180° \\ c &= 180° - 105° \\ &= 75°. \end{aligned}$$

Consider the quadrilateral BODC. The four angles in any quadrilateral add up to $360°$. So

$$\begin{aligned} b + c + 210° + 20° &= 360° \\ b &= 360° - 210° - 20° - c \\ &= 130° - c \\ &= 130° - 75° \\ &= 55°. \end{aligned}$$

Worked Example 3

In the diagram the line AB is a diameter and O is the centre of the circle. Find the angles marked.



Solution

Consider triangle OAC. Since OA and OC are radii, triangle OAC is isosceles. So

$$a = 50°.$$

The angles in a triangle add to $180°$, so for triangle OAC,

$$\begin{aligned} a + b + 50° &= 180° \\ b &= 180° - 50° - a \\ &= 80°. \end{aligned}$$

Since AB is a diameter of the circle, the angle ACB is a right angle, so

$$\begin{aligned} a + 20° + c &= 90° \\ c &= 90° - 20° - a \\ &= 20°. \end{aligned}$$

Angle d and angle OAC are angles in the same segment, so

$$\begin{aligned} d &= \text{angle OAC} \\ &= 50°. \end{aligned}$$

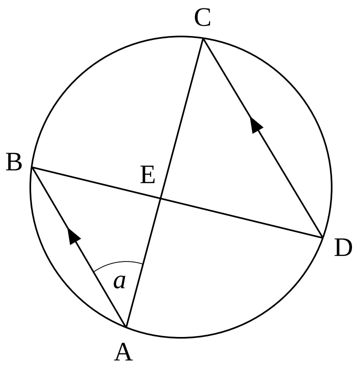
Angle e is drawn on the same arc as the angle at the centre, AOC, so

$$b = 2e$$

$$e = \frac{1}{2}b$$

$$= 40°.$$

Worked Example 4



In the diagram the chords AB and CD are parallel.
Prove that the triangles ABE and DEC are isosceles.

Solution

Angles a and BDC are angles in the same segment, so

$$\text{angle BDC} = a.$$

Since AB and DC are parallel, angles a and ACD are equal alternate angles,

$$\text{angle ACD} = a = \text{angle BDC}.$$

Hence in triangle DEC, the base angles at C and D are equal, so the triangle is isosceles.

The angle at B, angle ABD, equals the angle at C, angle ACD, because they are angles in the same segment:

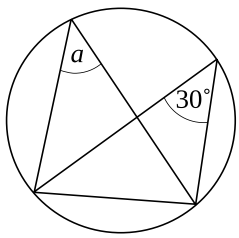
$$\text{angle ABD} = \text{angle ACD} = a.$$

Hence triangle ABE is isosceles, since the base angles at A and B are equal.

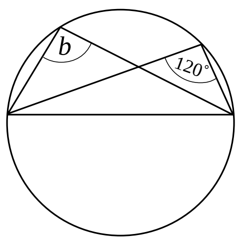
Exercises

1. Find all the angles marked with a letter in each of the following diagrams. In each case the centre of the circle is marked O. *Give reasons for your answers.*

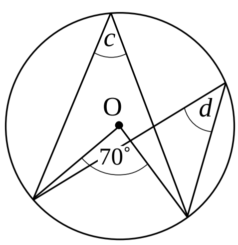
(a)



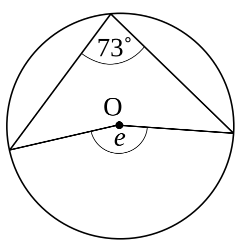
(b)



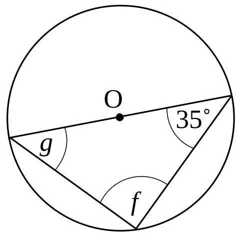
(c)



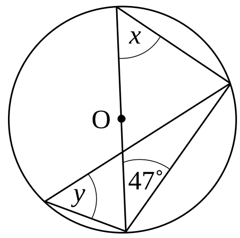
(d)



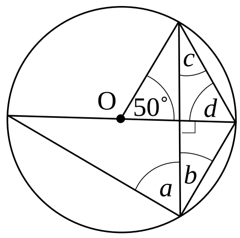
(e)



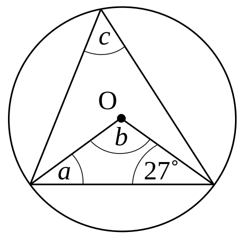
(f)



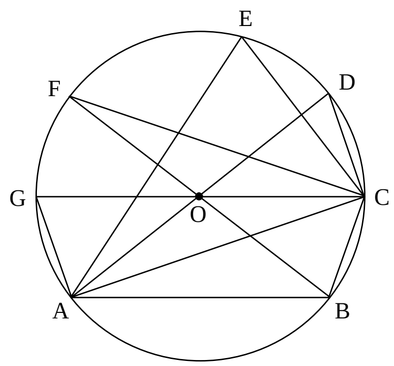
(g)



(h)



2. In the diagram, O is the centre of the circle and AOD, BOF and COG are diameters.



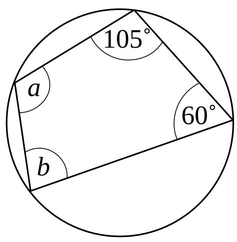
(a) Identify the equal angles.

(b) Identify the right angles.

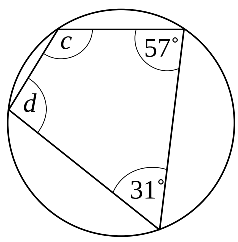
Give reasons for your answers.

3. Find all of the angles marked with a letter in each of the following diagrams.
Give reasons for your answers.

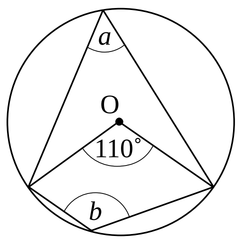
(a)



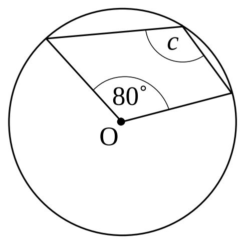
(b)



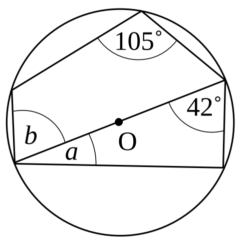
(c)



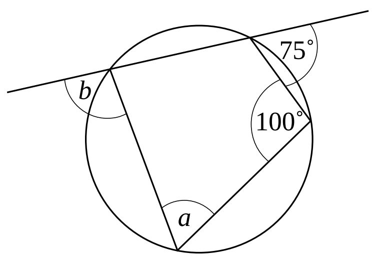
(d)



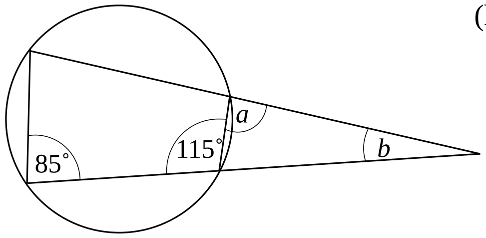
(e)



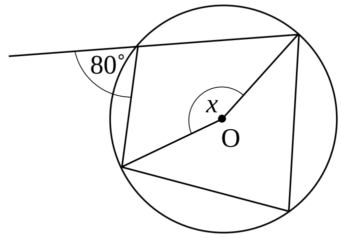
(f)



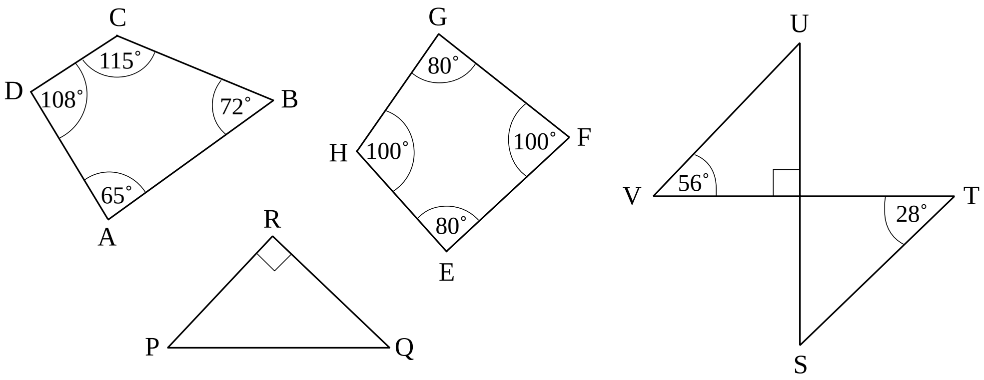
(g)



(h)

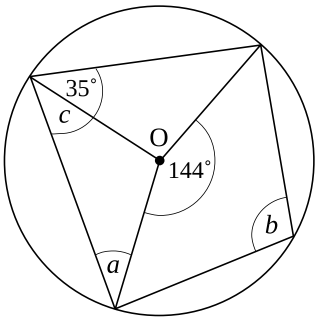


4. Which of the following points are concyclic points?

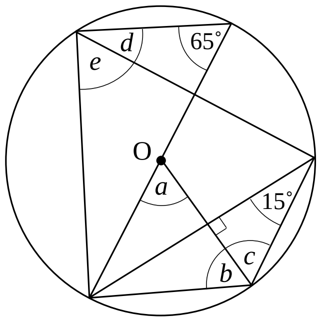


5. Find all the angles marked with a letter in the following diagrams. In each case, the point O is the centre of the circle.

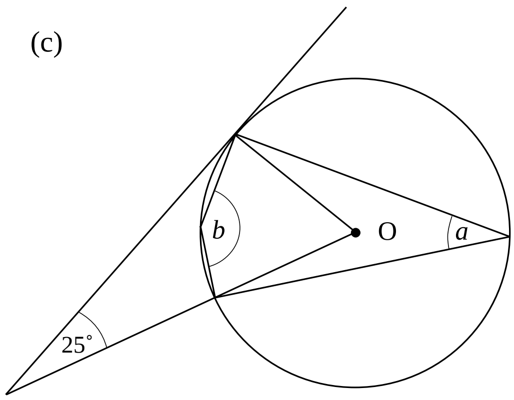
(a)



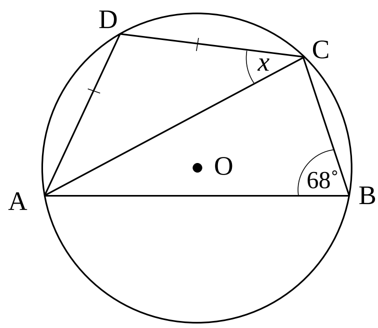
(b)



(c)



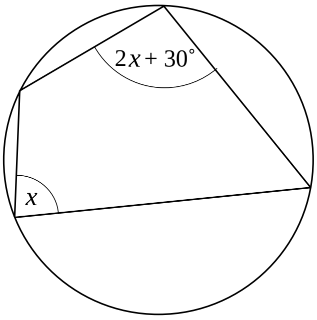
(d)



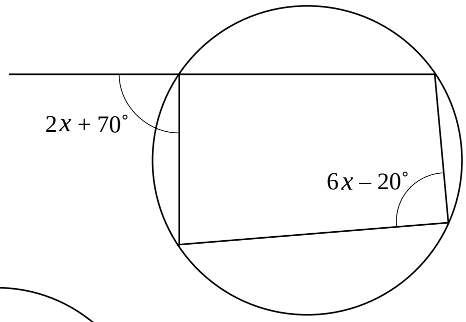
AD = DC

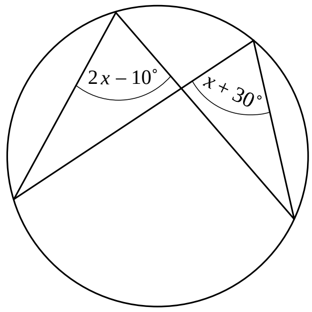
6. Find the value of x in each of the following diagrams.

(a)



(b)

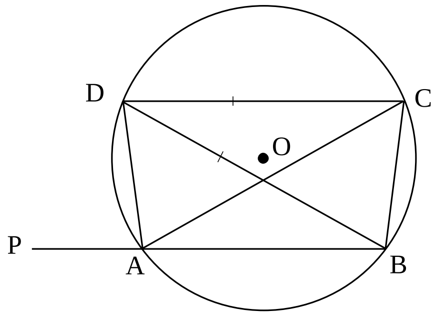




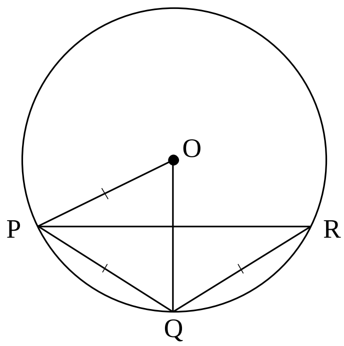
(c)

7. In the diagram, O is the centre of the circle, BD = DC and PAB is a straight line.

Prove that AD bisects the angle CAP.



8.



In the diagram O is the centre of the circle,

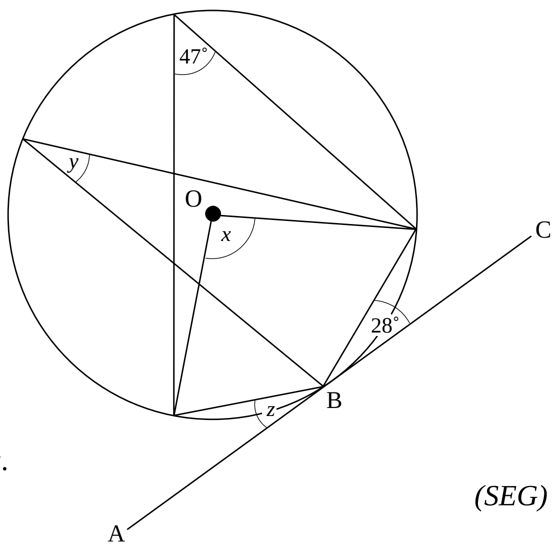
OP = PQ = QR

Prove that OP and QR are parallel lines.

9. O is the centre of the circle.

ABC is a tangent to the circle at B.

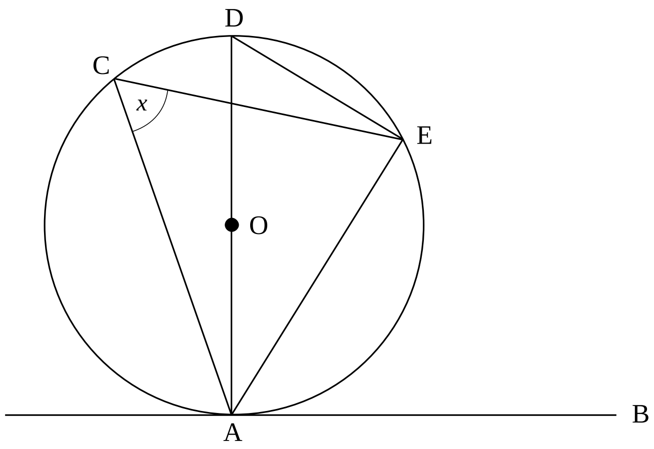
Not to scale



Work out the size of angles x, y and z.

(SEG)

10.



In the diagram, O is the centre of the circle, AD is a diameter and AB is a tangent.

Angle ACE = $x°$.

Find, in terms of x, the size of:

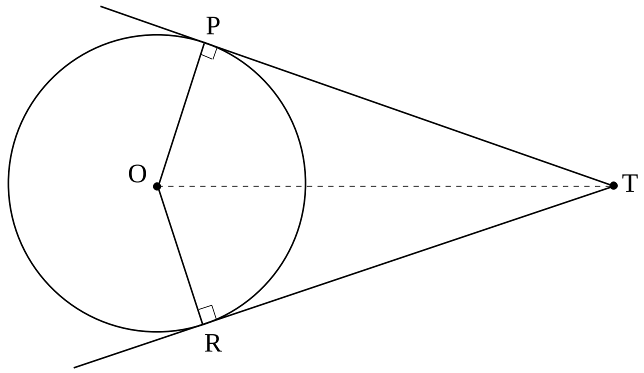
(a) angle ADE (b) angle DAE (c) angle EAB (d) angle AOE

(MEG)

3.10 Circles and Tangents

Some important results are stated below.

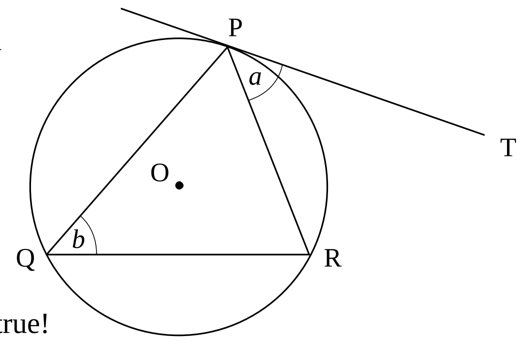
1. If two tangents are drawn from a point T to a circle with centre O, and P and R are the points of contact of the tangents with the circle, then, using symmetry,



(a) PT = RT

(b) Triangles TPO and TRO are congruent.

2. The angle between a tangent and a chord equals an angle on the circumference subtended by the same chord; e.g. $a = b$ in the diagram.



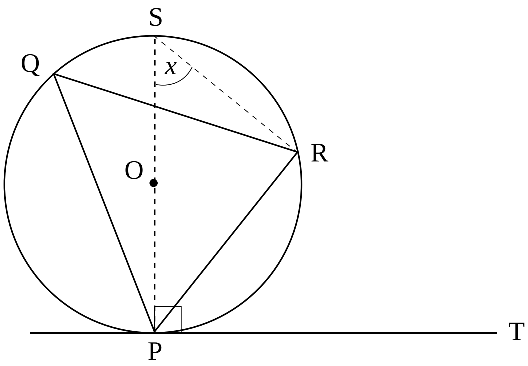
This is known as the

alternate segment theorem

and needs a proof, as it is not obviously true!

Proof

Construct the diameter POS, as shown.



We know that

angle SRP = 90°

since PS is a diameter.

Now

angle PSR = angle PQR = x°, say,

so

$$\text{angle SPR} = 180° - 90° - x$$
$$= 90° - x°.$$

But

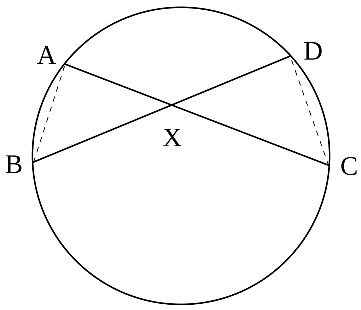
$$\text{angle RPT} = 90° - (\text{angle SPR})$$
$$= 90° - (90° - x°)$$
$$= x°$$
$$= \text{angle PQR}$$

and the result is proved.

3. For any two intersecting chords, as shown,

$$\boxed{\text{AX . CX = BX . DX}}$$

The proof is based on similar triangles.



Proof

In triangles AXB and DXC,

angle BAC = angle BDC (equal angles subtended by chord BC)

and

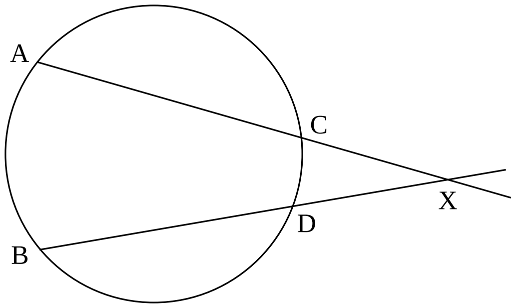
angle ABD = angle ACD (equal angles subtended by chord AD)

As AXB and DXC are similar,

$$\frac{\text{AX}}{\text{BX}} = \frac{\text{DX}}{\text{CX}} \quad \Rightarrow \quad \text{AX.CX} = \text{BX.DX}$$

as required.

This result will still be true even when the chords intersect *outside* the circle, as illustrated opposite.



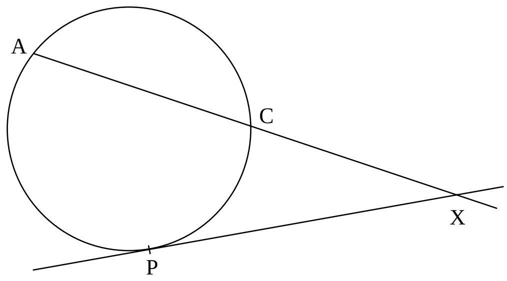
How can this be proved?

When the chord BD becomes a tangent, and B and D coincide at the point P, then

$$AX . CX = PX . PX$$

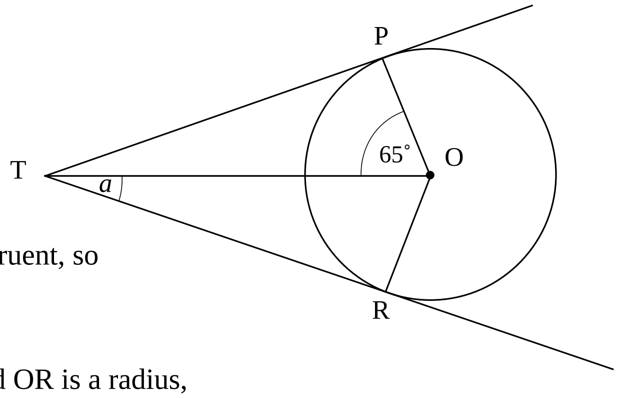
or

$$\boxed{AX . CX = PX^2}$$



Worked Example 1

Find the angle a in the diagram.



Solution

The triangles TOR and TOP are congruent, so

$$\text{angle TOR} = 65°.$$

Since TR is a tangent to the circle and OR is a radius,

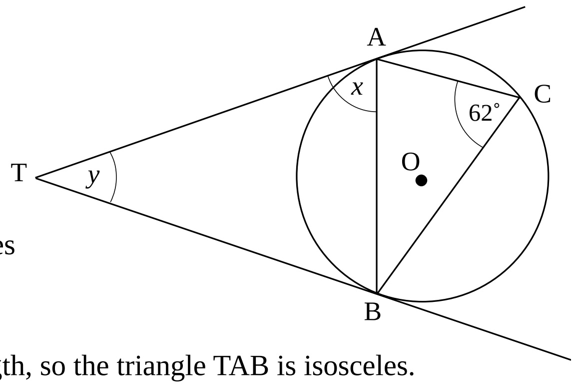
$$\text{angle TRO} = 90°.$$

Hence

$$\begin{aligned} a &= 180° - 90° - 65° \\ &= 25°. \end{aligned}$$

Worked Example 2

Find the angles x and y in the diagram.



Solution

The alternate angle segment theorem gives

$$x = 62°.$$

The tangents TA and TB are equal in length, so the triangle TAB is isosceles.
So

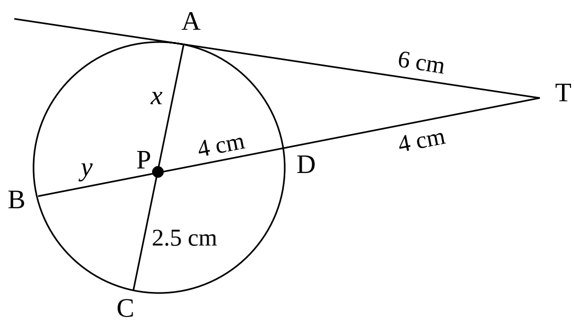
$$\text{angle ABT} = x = 62°.$$

Hence

$$\begin{aligned} y + 62° + 62° &= 180° \quad \text{(the angles in triangle TAB add up to } 180°\text{)} \\ y &= 56°. \end{aligned}$$

Worked Example 3

Find the unknown lengths in the diagram.



Solution

Since AT is a tangent,

$$\begin{aligned} AT^2 &= BT . DT \\ 36 &= BT \times 4 \\ BT &= 9. \end{aligned}$$

Hence

$$y + 8 = BT = 9$$
$$y = 1 \text{ cm}.$$

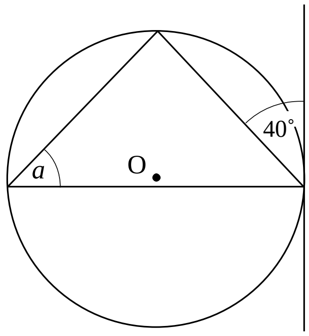
AC and BD are intersecting chords, so

$$AP.PC = BP.PD$$
$$2.5x = 1 \times 4$$
$$x = \frac{4}{2.5}$$
$$= 1.6 \text{ cm}.$$

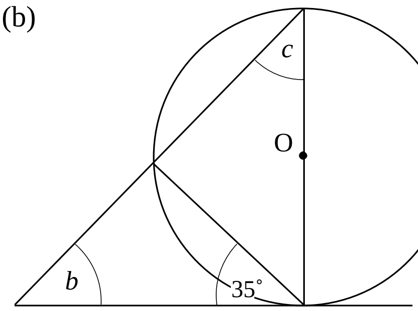
Exercises

1. Find the angles marked in the diagrams. In each case O is the centre of the circle.

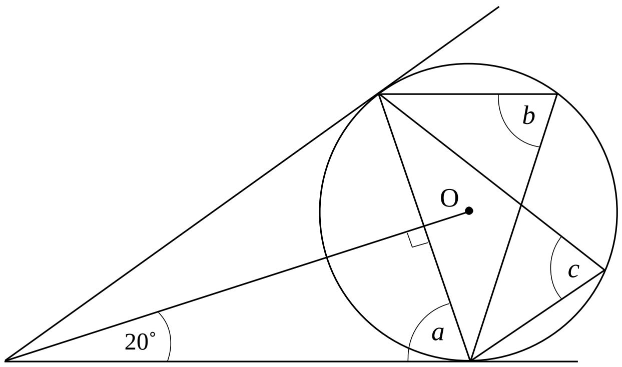
(a)



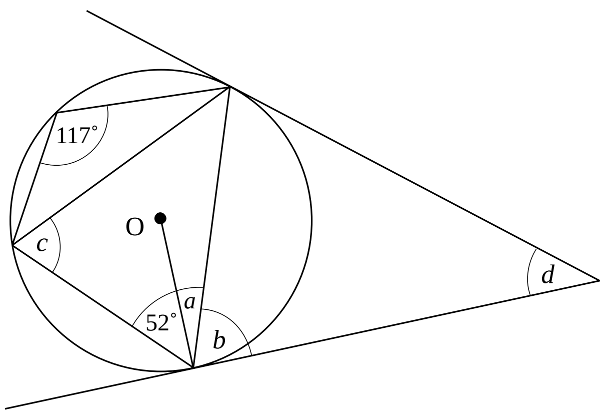
(b)



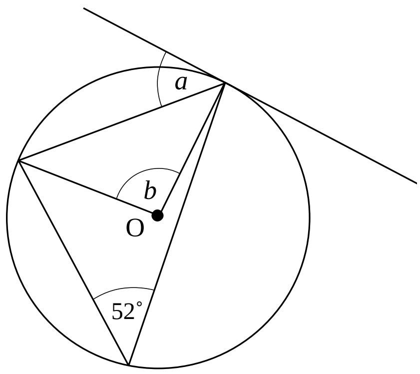
(c)



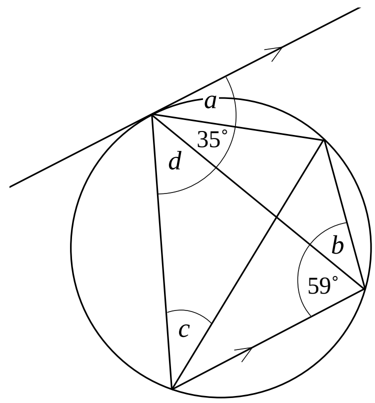
(d)



(e)

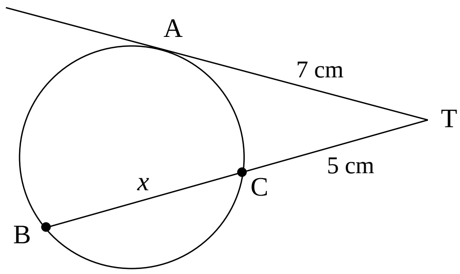


(f)

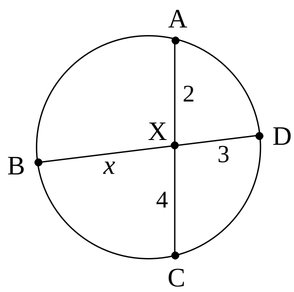


2. Find the unknown lengths in the following diagrams.

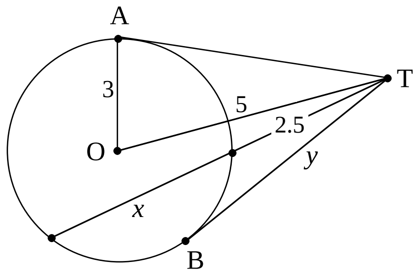
(a)



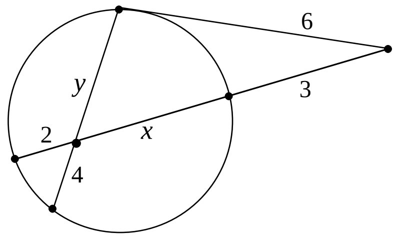
(b)



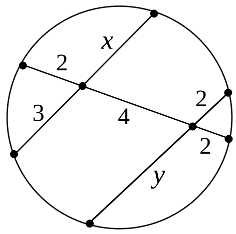
(c)



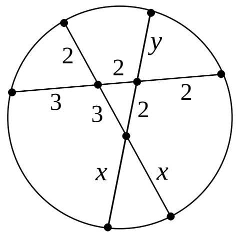
(d)



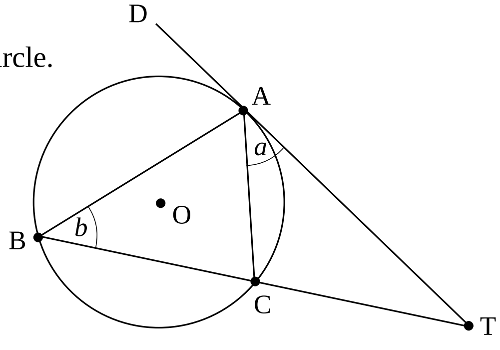
(e)



(f)



3. In the diagram, TAD is a tangent to the circle.



(a) Prove that $a = b$.

(b) Show that triangles BTA and ACT are similar triangles.

(c) If

$$BC = 5 \text{ cm}$$
$$CT = 4 \text{ cm}$$

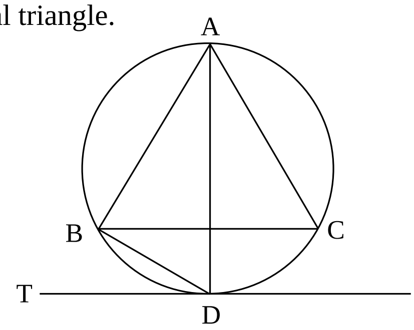
calculate the length of the tangent AT.

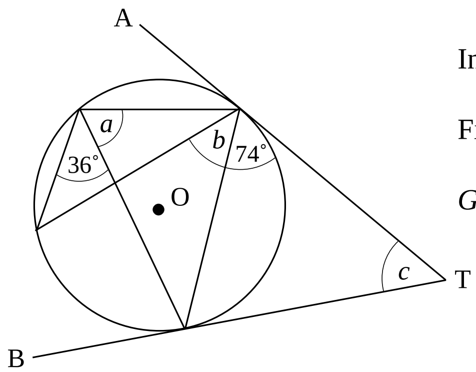
4. The triangle ABC in the diagram is an equilateral triangle.

The line AD bisects angle BAC.

(a) Prove that the line AD is a diameter of the circle.

(b) Hence find the angle BDT, where DT is a tangent to the circle.



5. 

In the diagram, TA and TB are tangents.

Find the angles a, b and c.

Give reasons for your answers.

6. AT and BT are tangents to the circle, centre C.

P is a point on the circumference, as shown.

Angle BAT = 65°.

Calculate the size of

(a) x (b) y (c) z .