AQA Approved

AQA
Level 2

Certificate in Further Mathematics

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Introduction

This book has been written to support the AQA Level 2 Certificate in Further Mathematics, but you may also use it independently as an introduction to Mathematics beyond GCSE. It is expected that many of the students using this book could be working with little day-to-day teacher support and with this in mind the text has been written in an interactive way and the answers are fuller than is often the case in books of this nature.

The qualification is designed for high-achieving students who have already acquired, or are expected to achieve, grades 7 to 9 in GCSE Mathematics. It is hoped that many of these students will progress to study Mathematics at Advanced Level and beyond.

Higher order mathematical skills are studied in greater depth with an emphasis on algebraic reasoning, rigorous argument and problem-solving skills. Students following this course will be well prepared to tackle a Level 3 Mathematics qualification.

The content is split into Algebra, Geometry, Calculus and Matrices, with each section containing work that stretches and challenges, and which goes beyond the Key Stage 4 Programme of Study. The topics are frequently linked together as progress is made through the book, highlighting the beauty and inter-connectedness of mathematics.

Each chapter begins with a quote, designed to engage and bring the topic to life and/or provide an alternative viewpoint. The chapters are then broken down into sub-sections, each with a short introduction followed by a number of worked examples (with solutions) covering important techniques and question styles, and finally one or more sets of exercise questions. Coloured boxes, hints and notes help to clarify some of the key points.

In addition, each chapter includes a number of activities. These are often used to introduce a new concept, or to reinforce the examples in the text. Throughout the book the emphasis is on understanding the mathematics being used rather than merely being able to perform the calculations, but the exercises do, nonetheless, provide plenty of scope for practising basic techniques.

Three symbols are used throughout the book:

! This 'warning sign' alerts you either to restrictions that need to be imposed or to possible pitfalls.

PS This indicates a problem-solving question. These questions will sometimes involve more than one topic area.

RWC This indicates a question that relates to real-world contexts.

Numerous 'Discussion points' are used throughout as prompts to help you understand the theory that has been, or is about to be, introduced. Answers to these are also included.

'Prior knowledge' boxes highlight the GCSE Mathematics, or content earlier in the book, that you should be familiar with before you tackle a topic.

'Future uses' sections explain how the mathematics covered in a chapter can be used for further study, including at later points in the book, while 'Real-world contexts' explain the applications of the mathematics covered in each chapter. Also at the end of each chapter you will find a list of learning outcomes and key points.

A short glossary of key words is provided, followed by two practice question papers. Answers to these, all exercise questions, activities and discussion points are then given at the back of the book.

It is hoped that students who use this book will develop a fascination for mathematics, be inspired and challenged by the rigorous nature of the course and be able to appreciate the power of mathematics for its own sake as well as a problem-solving tool.

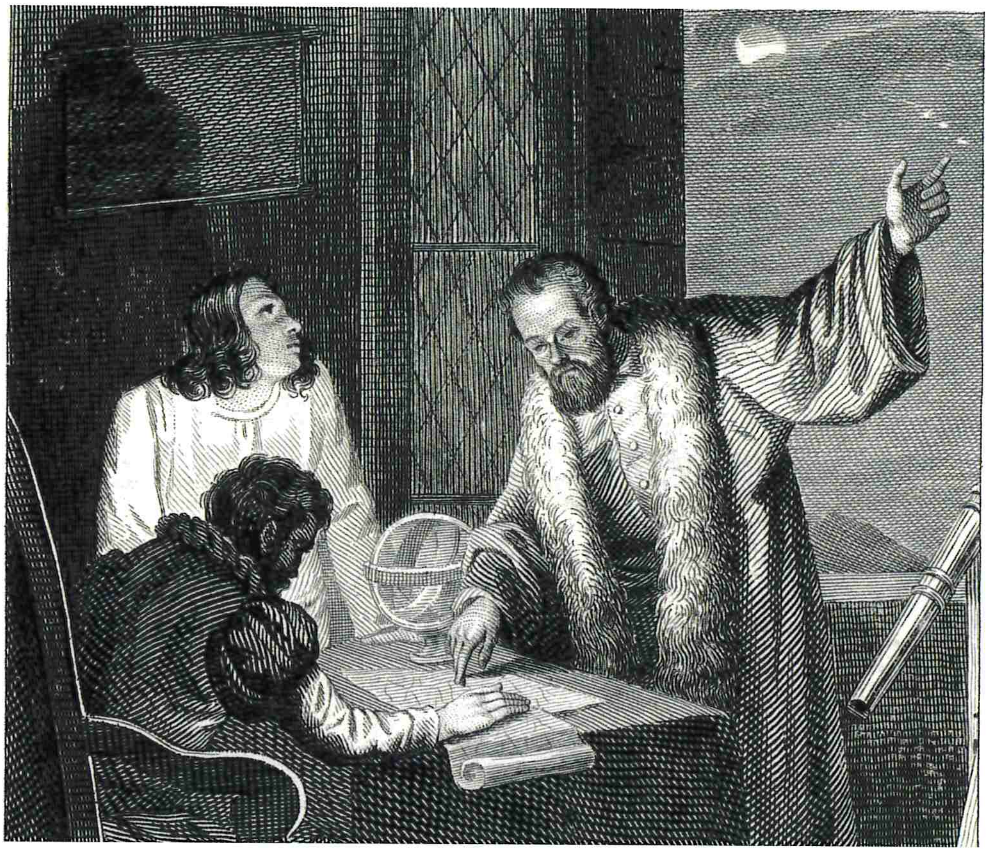
This book is supported by free narrated step-by-step examples, which you can access online at:

www.hoddereducation.co.uk/aqal2maths

Each example is numbered to correspond to the topics in the textbook. This makes it quick and easy for you to find additional guidance for each topic that will help you to approach the questions in the book.

1

Number and algebra I



The Book of Nature is written in the language of mathematics

Galileo Galilei

Prior knowledge

Students are expected to be familiar with all number and ratio topics from GCSE. In Exercise 1A, the GCSE specification references particularly assessed are N2, N3, N8, N12, N15, R4, R5 and R9.

1 Numbers and the number system

Number will be tested implicitly throughout the course. The following examples and questions provide practice of some of the basic number skills that may be needed.

Example 1.1

Simplify the ratio 3 kilometres : 840 metres.

Solution

$$\begin{aligned} 3\,\text{km} : 840\,\text{m} &= 3000\,\text{m} : 840\,\text{m} \\ &= 3000 : 840 \\ &= 300 : 84 \\ &= 100 : 28 \\ &= 25 : 7 \end{aligned}$$

Example 1.2

Work out 43% of 5680

Solution

$$\begin{aligned}43\% \text{ of } 5680 &= 0.43 \times 5680\\ &= 2442.4\end{aligned}$$

Example 1.3

Increase 540 by 17.5%.

Solution

$$\begin{aligned}540 + 17.5\% \text{ of } 540 &= 540 \times 117.5\%\\ &= 540 \times 1.175\\ &= 634.5\end{aligned}$$

Example 1.4

Without using a calculator, work out $\frac{9}{10} - \frac{2}{5} \div \frac{6}{7}$.

Solution

$$\begin{aligned}\frac{9}{10} - \frac{2}{5} \div \frac{6}{7} &= \frac{9}{10} - \frac{2}{5} \times \frac{7}{6}\\ &= \frac{9}{10} - \frac{2 \times 7}{5 \times 6}\\ &= \frac{9}{10} - \frac{1 \times 7}{5 \times 3}\\ &= \frac{9}{10} - \frac{7}{15}\\ &= \frac{27}{30} - \frac{14}{30}\\ &= \frac{13}{30}\end{aligned}$$

Example 1.5

Given the ratios $x : y = 5 : 3$ and $y : z = 4 : 7$, work out the ratio $x : z$ in its simplest form.

Solution

$x : y = 20 : 12$ and $y : z = 12 : 21$

so $x : y : z = 20 : 12 : 21$

so $x : z = 20 : 21$

Example 1.6

Work out, giving your answer to 3 significant figures, $\dfrac{3.76 \times 34}{78.4 \times 980}$.

Solution

$$\frac{3.76 \times 34}{78.4 \times 980} = 1.663\,890\,046 \times 10^{-3}$$

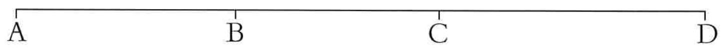
$$= 0.001\,663\,890\,046$$

$$= 0.00166 \quad (3 \text{ s.f.})$$

Exercise 1A

Do not use a calculator for the questions marked ⋆.

① ABCD is a straight line (not drawn to scale). AB = 4 cm, AC = 10 cm, AD = 22 cm.



Work out these ratios, giving your answers in their simplest form.

(i) AC : AB (ii) AB : BC (iii) AD : AB

(iv) BC : CD (v) BD : BC

> ! If a question involving money requires an answer to be given in pounds and pence, remember to give any non-integer answers to 2 decimal places.

⋆② Work out

(i) 60% of £115 (ii) $33\frac{1}{3}$% of 780 (iii) 17.5% of 64 cm.

③ Work out

(i) 95% of 7540 (ii) $12\frac{1}{2}$% of 53.76 (iii) 4.2% of £150.

⋆④ (i) Increase 80 by 5%. (ii) Increase £240 by 75%.

(iii) Decrease £20 by 40%. (iv) Decrease 36 by $66\frac{2}{3}$%.

⑤ (i) Increase 650 by 14%. (ii) Decrease 3250 by 3.5%.

(iii) Decrease £3650 by 64%. (iv) Increase £46.30 by $5\frac{1}{2}$%.

⋆⑥ Work out, giving your answers as fractions in their simplest form,

(i) $\frac{3}{5} + \frac{2}{3} \times \frac{5}{6}$ (ii) $\left(\frac{1}{2}\right)^3 \div 4$ (iii) $3\frac{2}{5} - \frac{3}{4}$.

⑦ (i) Work out, giving your answer to 3 significant figures, $52.7 \div 4.93$

(ii) Work out, giving your answer to 2 significant figures, $5.9 - 0.53 \times 1.8$

(iii) Work out, giving your answer to 1 significant figure, $0.23 \times 0.14 + 0.09^2$

(iv) Work out, giving your answer to 2 decimal places, $\dfrac{19 + 36}{144 - 52}$.

⑧ A bag contains blue, green and white beads.

The ratio of blue beads to green beads is 4 : 3.

The ratio of green beads to white beads is 2 : 7.

Work out the smallest possible number of beads in the bag.

⑨ 55% of teachers in a school are female. The other 36 teachers are male. Work out the number of teachers in the school.

Prior knowledge

Students are expected to be familiar with all aspects of GCSE algebra. Exercise 1B particularly assesses GCSE specification reference A4.

2 Simplifying expressions

When you are asked to *simplify* an algebraic expression you need to write it in its most compact form. This will involve techniques such as collecting like terms, removing brackets, factorising and finding a common denominator (if the expression includes fractions).

Example 1.7

Simplify this expression.

$3a + 4b - 2c + a - 3b - c$

Solution

Expression $= 3a + a + 4b - 3b - 2c - c$ (collecting like terms)

$= 4a + b - 3c$

Example 1.8

Simplify this expression.

$2(3x - 4y) - 3(x + 2y)$

! A common error in questions like this is to forget to multiply all terms in the second bracket by −3.

Solution

Expression $= 6x - 8y - 3x - 6y$ (removing the brackets)

$= 3x - 14y$

$-3 \times 2y = -6y$

Example 1.9

Simplify this expression.

$3x^2yz \times 2xy^3$

Solution

Expression $= (3 \times 2) \times (x^2 \times x) \times (y \times y^3) \times z$ (collecting like terms)

$= 6x^3y^4z$

Example 1.10

Simplify this expression.

$$\frac{12a^3b^2c^2}{8ab^5c}$$

Solution

Divide the numerator and denominator by their highest common factor.

$$\frac{12a^3b^2c^2}{8ab^5c} = \frac{3a^{3-1}c^{2-1}}{2b^{5-2}}$$

$$= \frac{3a^2c}{2b^3}$$

Note

It is not necessary to include the intermediate step shown here.

Example 1.11

Factorise this expression.

$3a^2b + 6ab^2$

Discussion point

➜ Explain what the word *factorise* means.

Solution

First write the highest common factor of the two terms, and then work on the contents of the brackets.

$3a^2b + 6ab^2 = 3ab(a + 2b)$ ← $3a^2b = 3ab \times a$ and $6ab^2 = 3ab \times 2b$

Example 1.12

Simplify this expression.

$$\frac{2x^2}{3yz} \div \frac{4xy^2}{5z^2}$$

Solution

$$\begin{aligned} \text{Expression} &= \frac{2x^2}{3yz} \times \frac{5z^2}{4xy^2} \\ &= \frac{10x^2z^2}{12xy^3z} \\ &= \frac{5xz}{6y^3} \end{aligned}$$

Example 1.13

Write as a single fraction

$$\frac{x}{4t} - \frac{2y}{5t} + \frac{z}{2t}.$$

Solution

$$\begin{aligned} \frac{x}{4t} - \frac{2y}{5t} + \frac{z}{2t} &= \frac{5x}{20t} - \frac{8y}{20t} + \frac{10z}{20t} \\ &= \frac{5x - 8y + 10z}{20t} \end{aligned}$$

← $20t$ is the lowest common multiple of $4t$, $5t$ and $2t$.

Exercise 1B

① Simplify the following expressions.

(i) $12a + 3b - 7c - 2a - 4b + 5c$

(ii) $4x - 5y + 3z + 2x + 2y - 7z$

(iii) $3(5x - y) + 4(x + 2y)$

(iv) $2(p + 5q) - (p - 4q)$

(v) $x(x + 3) - x(x - 2)$

(vi) $a(2a + 3) + 3(3a - 4)$

(vii) $3p(q - p) - 3q(p - q)$

(viii) $5f(g + 2h) - 5g(h - f)$

② Factorise the following expressions by taking out the highest common factor.

(i) $8 + 10x^2$
(ii) $6ab + 8bc$
(iii) $2a^2 + 4ab$
(iv) $pq^3 + p^3q$
(v) $3x^2y + 6xy^4$
(vi) $6p^3q - 4p^2q^2 + 2pq^3$
(vii) $15lm^2 - 9l^3m^3 + 12l^2m^4$
(viii) $84a^5b^4 - 96a^4b^5$

③ Simplify the following expressions and factorise the results.

(i) $4(3x + 2y) + 8(x - 3y)$
(ii) $x(x - 2) - x(x - 8) + 6$
(iii) $x(y + z) - y(x + z)$
(iv) $p(2q - r) + r(p - 2q)$
(v) $k(l + m + n) - km$
(vi) $a(a - 2) - a(a + 4) + 2(a - 4)$
(vii) $3x(x + y) - 3y(x - 2y)$
(viii) $a(a - 2) - a(a - 4) + 8$

④ Simplify the following expressions as much as possible.

(i) $2a^2b \times 5ab^3$
(ii) $6p^3q \times 2q^3r$
(iii) $lm \times mn \times np$
(iv) $3r^3 \times 6s^2 \times 2rs$
(v) $ab \times 2bc \times 4cd \times 8de$
(vi) $3xy^2 \times 4yz^2 \times 5x^2z$
(vii) $2ab^3 \times 6a^4 \times 7b^6$
(viii) $6p^2q^3r \times 7pq^5r^4$

⑤ Simplify the following fractions as much as possible.

(i) $\dfrac{4a^2b}{2ab}$
(ii) $\dfrac{p^2}{q} \times \dfrac{q^2}{p}$
(iii) $\dfrac{8a}{3b^2} \times \dfrac{6b^3}{4a^2}$
(iv) $\dfrac{3ab}{2c^2} \times \dfrac{4cd}{6a^2}$
(v) $\dfrac{8xy^3z^2}{12yz}$
(vi) $\dfrac{3a^2}{9b^3} \div \dfrac{2a^4}{15b}$
(vii) $\dfrac{5p^3q}{8rs^2} \div \dfrac{15pq^5}{28r^4}$

⑥ Write the following expressions as single fractions.

(i) $\dfrac{2a}{3} + \dfrac{a}{4}$
(ii) $\dfrac{2x}{5} - \dfrac{x}{2} + \dfrac{3x}{4}$
(iii) $\dfrac{4p}{3} - \dfrac{3p}{4}$
(iv) $\dfrac{2s}{5} - \dfrac{s}{3} + \dfrac{4s}{15}$
(v) $\dfrac{3b}{8} - \dfrac{b}{6} + \dfrac{5b}{24}$
(vi) $\dfrac{3a}{b} - \dfrac{2a}{3b}$
(vii) $\dfrac{5}{2p} - \dfrac{3}{2q}$
(viii) $\dfrac{2x}{3y} - \dfrac{3x}{2y}$

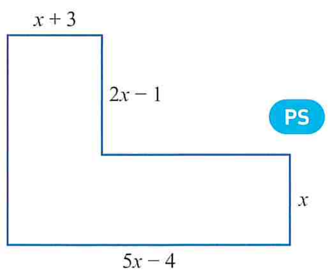


Figure 1.1

PS ⑦ The angles of the hexagon in Figure 1.1 are all 90° or 270°.

Its side lengths are given in terms of x.

(i) Work out its perimeter in terms of x.
(ii) Work out its area in terms of x.

Give your answers in simplified form.

PS ⑧ The rectangle in Figure 1.2 has length $5x + 2$ and width $3x - 1$.

Squares of side x are removed from each corner of the rectangle.

(i) Write down a simplified expression for the perimeter of the new shape.
(ii) Write down a simplified expression for the area of the new shape.

The new shape is the net of an open cuboid.

(iii) Write down an expression for the volume of the cuboid.

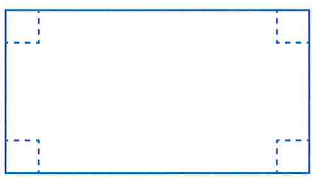


Figure 1.2

Discussion points

- What is an equation?
- What does solving an equation mean?

3 Solving linear equations

Linear equations can be solved in different ways. The final answer will always be the same regardless of the method used. The only rule to remember is that whatever is done to one side is also done to the other side. The following examples illustrate this. With practice you would probably omit some of the working.

Prior knowledge

Students are expected to be familiar with solving linear equations. Exercise 1C particularly assesses GCSE specification references A17 and A21.

Example 1.14

Solve this equation.

$3(3x - 17) = 2(x - 1)$

Solution

Multiply out the brackets	$\Rightarrow$	$9x - 51$	$= 2x - 2$
Subtract $2x$ from both sides	$\Rightarrow$	$9x - 51 - 2x$	$= 2x - 2 - 2x$
Tidy up	$\Rightarrow$	$7x - 51$	$= -2$
Add 51 to both sides	$\Rightarrow$	$7x - 51 + 51$	$= -2 + 51$
Tidy up	$\Rightarrow$	$7x$	$= 49$
Divide both sides by 7	$\Rightarrow$	x	$= 7$

Example 1.15

Solve this equation.

$\frac{1}{2}(x + 8) = 2x + \frac{1}{3}(4x - 5)$

Solution

Start by clearing the fractions by multiplying both sides by 6 (the least common multiple of 2 and 3).

Multiply both sides by 6	$\Rightarrow$	$6 \times \frac{1}{2}(x + 8) = 6 \times 2x + 6 \times \frac{1}{3}(4x - 5)$
Tidy up	$\Rightarrow$	$3(x + 8) = 12x + 2(4x - 5)$
Multiply out the brackets	$\Rightarrow$	$3x + 24 = 12x + 8x - 10$
Tidy up	$\Rightarrow$	$3x + 24 = 20x - 10$
Subtract $3x$ from both sides	$\Rightarrow$	$24 = 17x - 10$
Add 10 to both sides	$\Rightarrow$	$34 = 17x$
Divide both sides by 17	$\Rightarrow$	$x = 2$

Discussion point

- Why have the letter and the number swapped sides on the last line?

Solving linear equations

Sometimes you will need to set up the equation as well as solve it. When you are doing this, make sure that you define any variables you introduce.

Example 1.16

In a triangle, the largest angle is nine times as big as the smallest. The third angle is 60°.

(i) Write this information in the form of an equation.

(ii) Solve the equation to work out the sizes of the three angles.

Solution

(i) Let s = the smallest angle in degrees

So $9s$ = the largest angle

The sum of all three angles is 180°

$s + 9s + 60 = 180$

(ii) Solving $\Rightarrow 10s = 120$

$\Rightarrow \quad s = 12$

The largest angle is then $9 \times 12 = 108$

So the angles are 12°, 60° and 108°

Exercise 1C

① Solve the following equations.

(i) $2x - 3 = x + 4$

(ii) $5a + 3 = 2a - 3$

(iii) $2(x + 5) = 14$

(iv) $7(2y - 5) = -7$

(v) $5(2c - 8) = 2(3c - 10)$

(vi) $3(p + 2) = 4(p - 1)$

(vii) $3(2x - 1) = 6(x + 2) + 3x$

(viii) $\frac{x}{3} + 7 = 5$

(ix) $\frac{5y - 2}{11} = 3$

(x) $\frac{k}{2} + \frac{k}{3} = 35$

(xi) $\frac{2t}{3} - \frac{3t}{5} = 4$

(xii) $\frac{5p - 4}{6} - \frac{2p + 3}{2} = 7$

(xiii) $p + \frac{1}{3}(p + 1) + \frac{1}{4}(p + 2) = \frac{5}{6}$

② The length, l metres, of a field is 80 m greater than the width. The perimeter is 600 m.

(i) Write this information in the form of an equation in l.

(ii) Work out the area of the field.

PS ③ Ben and Chris are twins and their brother Stephen is four years younger. The total of their three ages is 17 years.

(i) Write this information in the form of an equation in s, Stephen's age in years.

(ii) What are all their ages?

PS ④ In a multiple-choice examination of 20 questions, four marks are given for each correct answer and one mark is deducted for each wrong answer. There is no penalty for not attempting a question. A candidate attempts a questions and gets c correct.

(i) Write down, and simplify, an expression for the candidate's total mark in terms of a and c.

(ii) A candidate attempts three-quarters of the questions and scores 40. Write down, and solve, an equation for the number of correct answers.

PS ⑤ Chris is three times as old as his son, Joe, and in 12 years' time he will be twice as old as him.

(i) Given that Joe is j years old now, write an expression for Chris' age in 12 years' time.

(ii) Write down, and solve, an equation in j.

PS ⑥ A square has sides of length $2a$ metres, and a rectangle has length $3a$ metres and width 3 metres.

(i) Write down, in terms of a, the perimeter of the square.

(ii) Write down, in terms of a, the perimeter of the rectangle.

(iii) The perimeters of the square and the rectangle are equal. Work out the value of a.

PS ⑦ The sum of five consecutive numbers is equal to 105. Let m represent the middle number.

(i) Write down the five numbers in terms of m.

(ii) Form an equation in m and solve it.

(iii) What are the five consecutive numbers?

PS ⑧ One rectangle has a length of $(x + 2)$ cm and a breadth of 2 cm. Another rectangle, of equal area, has a length of 5 cm and a width of $(x - 3)$ cm.

(i) Write down an equation in x and solve it.

(ii) What is the area of each of the rectangles?

Discussion point

- → A large ice cream costs 40p more than a small one. Two large ice creams cost the same as three small ones. What is the cost of each size of ice cream?
- → This is an example of the type of question that you might find in a puzzle book or the puzzle section of a newspaper or magazine. How would you set about solving it?

Discussion point

- → You may think that the following question appears to be very similar to the one on the left. What happens when you try to solve it?
- → A large ice cream costs 40p more than a small one. Five small ice creams plus three large ones cost 80p less than three small ice creams plus five large ones. What is the cost of each size of ice cream?

4 Algebra and number

Some algebra questions will involve using number skills.

Example 1.17

a is 75% of b and $b : c = 3 : 2$

Show that $8a = 9c$.

Solution

a is 75% of b

$$a = \frac{75}{100}b$$

$$a = \frac{3}{4}b \quad ①$$

$b : c = 3 : 2$

$$\frac{b}{c} = \frac{3}{2}$$

$$b = \frac{3}{2}c \quad ②$$

Substitute ② in ①

$$a = \frac{3}{4} \times \frac{3}{2}c$$

$$a = \frac{9}{8}c$$

$$8a = 9c$$

Example 1.18

Write an expression for x increased by 13%.

Solution

$$x \text{ increased by } 13\% = x + \frac{13}{100}x$$

$$= 1.13x$$

Example 1.19

$p : q = 4 : 5$

Work out $p + 2q : 4q$, giving your answer in its simplest form.

Solution

Thinking in terms of parts:

p is 4 parts, q is 5 parts

$p + 2q$ is $4 + 2 \times 5 = 14$ parts

$4q$ is 20 parts

$p + 2q : 4q = 14 : 20$

$= 7 : 10$

An alternative solution is:

$$p : q = 4 : 5 \quad \Rightarrow \quad \frac{p}{q} = \frac{4}{5}$$

$$\Rightarrow \quad p = \frac{4}{5}q$$

$$p + 2q : 4q = \frac{4}{5}q + 2q : 4q$$

$$= \frac{14}{5}q : 4q$$

$$= \frac{14}{5} : 4$$

$$= 7 : 10$$

Exercise 1D

① Write expressions for the following, giving your answers in their simplest form.

(i) 30% of b (ii) y% of 450 (iii) c% of d

② 60% of p = 40% of q

Work out p as a percentage of q.

③ Write expressions for the following, giving your answers in their simplest form.

(i) a increased by 20% (ii) b increased by 5%

(iii) k decreased by 35% (iv) m decreased by 2%

④ a increased by 80% is equal to b increased by 50%.

Show that $\frac{b}{a} = 1.2$

PS ⑤ p increased by 25% is equal to q decreased by 25%.

Work out p as a percentage of q.

PS ⑥ $x : y = 2 : 3$ and $y : z = 4 : 9$

Work out $x : y : z$, giving your answer in its simplest form.

PS ⑦ $a : b = 5 : 2$

(i) Write a in terms of b.

(ii) Work out $2a + b : b$, giving your answer in its simplest form.

(iii) Work out $7a - 5b : 4a$, giving your answer in its simplest form.

PS ⑧ $m : n = 3 : 8$ and r is 20% of n.

Work out $m : r$.

PS ⑨ y is 20% greater than x.

w is 20% less than y.

Work out the ratio $w : x$ in its simplest form.

PS ⑩ p is m% greater than q.

p is m% less than r.

Work out the ratio $r : q$ in terms of m.

PS ⑪ The ratio of boys to girls in a room is 3 : 7

16 boys enter and 6 girls leave. The ratio is now 4 : 5

How many boys and how many girls are now in the room?

5 Expanding brackets

Prior knowledge

Students are expected to be familiar with multiplication of two or three linear expressions. Exercise 1E particularly assesses GCSE specification reference A4h.

Discussion point

→ Why is $(x + 5)(2x - 3)$ a quadratic expression?

An expression of the form $ax^2 + bx + c$ (where the coefficient of x is non-zero) is a quadratic in x.

For example,

$x^2 + 3$

a^2 (a quadratic expression in a),

$2y^2 - 3y + 5$ (a quadratic expression in y).

Example 1.20

Expand $(x + 5)(2x - 3)$.

Solution

$$\begin{aligned}(x + 5)(2x - 3) &= x(2x - 3) + 5(2x - 3)\\ &= 2x^2 - 3x + 10x - 15\\ &= 2x^2 + 7x - 15\end{aligned}$$

This method has multiplied everything in the second bracket by each term in the first bracket. An alternative way of setting this out is used in the next example.

Example 1.21

Expand $(3x - 5)^2$.

Solution

$$(3x - 5)^2 = (3x - 5)(3x - 5)$$

Write the square as the product of two brackets so you don't forget the middle term.

$$\begin{array}{r} 3x - 5\\ \times\ 3x - 5\\ \hline -15x + 25\\ 9x^2 - 15x\quad\ \ \\ \hline 9x^2 - 30x + 25\end{array}$$

Multiply the top line by −5.

Multiply the top line by $3x$.

Add the two products.

Example 1.22

Multiply $(x^3 + 2x - 4)$ by $(x^2 - x + 3)$.

Solution

$$(x^3 + 2x - 4)(x^2 - x + 3)$$
$$= x^3(x^2 - x + 3) + 2x(x^2 - x + 3) - 4(x^2 - x + 3)$$
$$= x^5 - x^4 + 3x^3 + 2x^3 - 2x^2 + 6x - 4x^2 + 4x - 12$$
$$= x^5 - x^4 + 5x^3 - 6x^2 + 10x - 12$$

Example 1.23

Expand and simplify $(a - 2)^3$.

Solution

$(a - 2)^3 = (a - 2)(a - 2)^2$

First, work out $(a - 2)^2$

$$\begin{aligned}(a - 2)(a - 2) &= a(a - 2) - 2(a - 2)\\ &= a^2 - 2a - 2a + 4\\ &= a^2 - 4a + 4\end{aligned}$$

Then multiply this by $(a - 2)$

$$\begin{aligned}(a - 2)^3 &= (a - 2)(a^2 - 4a + 4)\\ &= a(a^2 - 4a + 4) - 2(a^2 - 4a + 4)\\ &= a^3 - 4a^2 + 4a - 2a^2 + 8a - 8\\ &= a^3 - 6a^2 + 12a - 8\end{aligned}$$

Exercise 1E

① Expand the following expressions.

(i) $(x + 5)(x + 4)$
(ii) $(x + 3)(x + 1)$
(iii) $(a + 5)(2a - 1)$
(iv) $(2p + 3)(3p - 2)$
(v) $(x + 3)^2$
(vi) $(2x + 3)(2x - 3)$
(vii) $(2 - 3m)(m - 4)$
(viii) $(6 + 5t)(2 - t)$
(ix) $(4 - 3x)^2$
(x) $(m - 3n)^2$

② (i) Multiply $(x^3 - x^2 + x - 2)$ by $(x^2 + 1)$
(ii) Multiply $(x^4 - 2x^2 + 3)$ by $(x^2 + 2x - 1)$
(iii) Multiply $(2x^3 - 3x + 5)$ by $(x^2 - 2x + 1)$
(iv) Multiply $(x^5 + x^4 + x^3 + x^2 + x + 1)$ by $(x - 1)$
(v) Expand $(x + 2)(x - 1)(x + 3)$
(vi) Expand $(2x + 1)(x - 2)(x + 4)$
(vii) Expand and simplify $(x + 1)^3$
(viii) Expand and simplify $(p - 5)^3$
(ix) Expand and simplify $(2a + 3)^3$
(x) Simplify $(2x^2 - 1)(x + 2) - 4(x + 2)^2$
(xi) Simplify $(x^2 - 1)(x + 1) - (x^2 + 1)(x - 1)$

Hint: Expand the first two sets of brackets first.

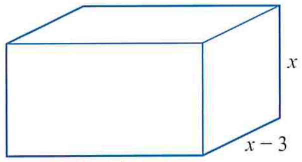


Figure 1.3

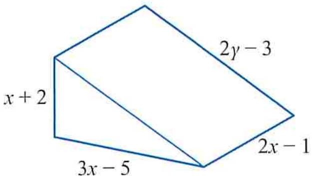


Figure 1.4

③ The cuboid in Figure 1.3 has length $(2x + 1)$, width $(x - 3)$, and height x.

(i) Work out its volume.

(ii) Work out its surface area.

Leave your answers in expanded and simplified form.

④ The prism in Figure 1.4 has three rectangular faces, and two congruent right-angled triangular faces. It has length $(3x - 5)$ and height $(x + 2)$. Its slant height is $(2y - 3)$ and its width is $(2x - 1)$.

(i) Work out its volume.

(ii) Work out its surface area.

Leave your answers in expanded and simplified form.

⑤ (i) Expand and simplify $(a + b)^2$.

(ii) Hence expand and simplify $(a + b)^3$.

(iii) Hence expand and simplify $(a + b)^4$.

(iv) Hence expand and simplify $(a + b)^5$.

⑥ Use your answers to question 5 to write down the expansions of

(i) $(x + 6)^3$

(ii) $(p - 2)^3$

(iii) $(2y + 1)^4$

(iv) $(x - 3)^4$

(v) $(3w - 4)^5$.

In the expansion of $(a + b)^3$ replace a with x, and b with 6.

ACTIVITY 1.1

Using your answers to Exercise 1E, question 5, make predictions about the simplified expansion of $(a + b)^6$.

(i) How many terms will there be in the simplified expansion?

(ii) What will be the coefficient of the a^6 term?

(iii) What will be the coefficient of the a^5b term in the simplified expansion?

(iv) Can you make any other predictions?

6 The binomial expansion

This section deals with the expansion of $(a + b)^n$ where n is a positive integer.

You already know
$$(a + b)^2 = (a + b)(a + b)$$
$$= a^2 + 2ab + b^2$$

Repeatedly multiplying by $(a + b)$ gives
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Each time you multiply by $(a + b)$, the simplified expansion gains an extra term.

So the simplified expansion of $(a + b)^n$ will have $n + 1$ terms.

Each term will be of the form $Pa^q b^r$, where the indices are non-negative integers whose sum is n.

The value of P is found in Pascal's triangle.

Pascal's triangle

Consider the coefficients of the expansions of $(a + b)^n$ and $(a + b)^{n+1}$.

For example:

$$\begin{aligned}(a+b)^5 &= (a+b)(a+b)^4\\ &= (a+b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)\\ &= 1a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + 1ab^4\\ &\quad + 1a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + 1b^5\\ &= 1a^5 + (4+1)a^4b + (6+4)a^3b^2 + (4+6)a^2b^3\\ &\quad + (1+4)ab^4 + 1b^5\\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

The coefficients of $(a + b)^5$ are the sums of adjacent coefficients of the $(a + b)^4$ expansion.

The coefficients of $(a + b)^n$ form Pascal's triangle:

$(a+b)^0$: 1

$(a+b)^1$: 1 1

$(a+b)^2$: 1 2 1

$(a+b)^3$: 1 3 3 1

$(a+b)^4$: 1 4 6 4 1

The 1 at the top of the triangle is usually referred to as the 0th row.

Figure 1.5

Each number in the triangle is the sum of the two numbers above it.

row 4 is 1 4 6 4 1

row 5 is 1 5 10 10 5 1

Figure 1.6

So $\quad (a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$

$\quad\quad = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Example 1.24

Expand $(2x + 3y)^3$.

Solution

In the expansion of $(a + b)^3$, replace a with $2x$ and b with $3y$.

$$\begin{aligned}(2x + 3y)^3 &= 1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3 \\ &= 1 \times 8x^3 \times 1 + 3 \times 4x^2 \times 3y + 3 \times 2x \times 9y^2 + 1 \times 1 \times 27y^3 \\ &= 8x^3 + 36x^2y + 54xy^2 + 27y^3\end{aligned}$$

Example 1.25

Expand $(3 - w)^4$.

Solution

In the expansion of $(a + b)^4$, replace a with 3 and b with $-w$.

$$\begin{aligned}(3 - w)^4 &= 1(3)^4(-w)^0 + 4(3)^3(-w)^1 + 6(3)^2(-w)^2 + 4(3)^1(-w)^3 + 1(3)^0(-w)^4 \\ &= 1 \times 81 \times 1 + 4 \times 27 \times (-w) + 6 \times 9 \times w^2 + 4 \times 3 \times (-w^3) + 1 \times 1 \times w^4 \\ &= 81 - 108w + 54w^2 - 12w^3 + w^4\end{aligned}$$

Note

It is not necessary to learn the expansions of $(a + b)^n$.
Instead, just learn one of the rows of Pascal's triangle.

Example 1.26

Expand $(x + 2)^5$.

Solution

The third row of Pascal's triangle is 1, 3, 3, 1.

Every row starts and finishes with 1, so the fourth row is 1, 1 + 3, 3 + 3, 3 + 1, 1 which simplifies to 1, 4, 6, 4, 1.

And the fifth row is 1, 1 + 4, 4 + 6, 6 + 4, 4 + 1, 1 which is 1, 5, 10, 10, 5, 1

So $\quad (x + 2)^5 = 1x^?2^? + 5x^?2^? + 10x^?2^? + 10x^?2^? + 5x^?2^? + 1x^?2^?$

One set of indices goes up from 0 to 5, whilst the other goes down from 5 to 0.

So $\quad \begin{aligned}(x + 2)^5 &= 1x^0 2^5 + 5x^1 2^4 + 10x^2 2^3 + 10x^3 2^2 + 5x^4 2^1 + 1x^5 2^0 \\ &= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5\end{aligned}$

Example 1.27

Work out the first three terms in ascending powers of x in the expansion of $(2 + 5x)^6$.

Solution

Row 5 of Pascal's triangle is 1 5 10 10 5 1

The first three numbers in row 6 are therefore 1 6 15

So the first three terms in the expansion of $(2 + 5x)^6$ are

$$1 \times 2^6 \times (5x)^0 \quad + \quad 6 \times 2^5 \times (5x)^1 \quad + \quad 15 \times 2^4 \times (5x)^2$$

$$= \quad 1 \times 64 \times 1 \quad + \quad 6 \times 32 \times 5x \quad + \quad 15 \times 16 \times 25x^2$$

$$= \quad 64 \quad + \quad 960x \quad + \quad 6000x^2$$

! After expanding an expression of the form $(a + b)^n$, and before simplifying, check that the sum of each pair of indices is n.

Exercise 1F

① Use Pascal's triangle to expand

(i) $(1 + x)^3$ (ii) $(y + 1)^4$ (iii) $(x + y)^5$

(iv) $(5 + w)^3$ (v) $(p + 4)^4$ (vi) $(2 + m)^5$.

② Use Pascal's triangle to expand

(i) $(x - y)^3$ (ii) $(1 - 2x)^4$ (iii) $(2 - y)^5$

(iv) $(5 - 2p)^3$ (v) $(3x - 4)^4$ (vi) $(4x - 1)^5$.

③ Work out the first three terms in ascending powers of x in the expansion of $(1 + x)^6$.

④ Work out the first three terms in descending powers of x in the expansion of $(2 + x)^7$.

⑤ Work out the coefficient of the x^3 term in the expansion of $(4 - 3x)^5$.

⑥ Write down the second number in the 10th row of Pascal's triangle.

⑦ Write down the last number in the 19th row of Pascal's triangle.

⑧ Write down the third number in the 9th row of Pascal's triangle.

⑨ Expand $\left(3x^2 + \frac{1}{x}\right)^4$.

⑩ (i) Expand $(1 + 2x)^5$.

(ii) Hence write down the expansion of $(1 - 2x)^5$.

(iii) Hence simplify $(1 + 2x)^5 - (1 - 2x)^5$.

⑪ (i) Expand $(3 + w)^3$.

(ii) Hence, by replacing w with $x + 2y$, write down the expansion of $(3 + x + 2y)^3$.

PS ⑫ The simplified expansion of $(mx + y)^n$ includes the term $240x^2y^4$.

(i) Write down the value of n.

(ii) Hence work out the possible values of m.

(iii) Hence work out the coefficient of the x^4y^2 term.

PS ⑬ In the expansion of $\left(x + \frac{2}{x}\right)^6$ work out the term which is independent of x.

PS ⑭ Given that the 10th row of Pascal's triangle is

1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1,

work out the coefficient of x^2 in the expansion of $\left(x - \frac{2}{x}\right)^{10}$.

REAL-WORLD CONTEXT

The binomial expansion has many applications in the real world, including in the distribution of IP addresses. It is also used by economists when making predictions for the future behaviour of markets, and similarly by meteorologists when forecasting the weather.

ACTIVITY 1.2

(i) How are the numbers 1, 11, 121, 1331, 14641 related?

(ii) Write a formula for the sum of the numbers in the nth row of Pascal's triangle. (Assume that the top row, which contains only '1', is the 0th row.)

(iii) Write a formula for the second number in the nth row of Pascal's triangle.

(iv) Write a formula for the third number in the nth row of Pascal's triangle.

(v) Each row of Pascal's triangle reads the same backwards as forwards.

What single word describes such a property?

FUTURE USES

At A-Level, the factorial function is used to quickly generate the numbers of Pascal's triangle. Students may like to investigate the ${}_nC_r$ function on their calculator to see how it relates to this topic.

Prior knowledge

Students are expected to be familiar with the manipulation of surd expressions. Exercise 1G particularly assesses GCSE specification reference N8h.

7 Manipulating surds

Simplifying expressions containing square roots

In mathematics there are times when it is helpful to be able to manipulate square roots, rather than just find their values from a calculator.
This ensures that you are working with the exact value, not just a rounded version.

Example 1.28

Simplify the following.

(i) $\sqrt{8}$

(ii) $\sqrt{6} \times \sqrt{3}$

(iii) $\sqrt{32} - \sqrt{18}$

(iv) $(4 + \sqrt{3})(4 - \sqrt{3})$

Solution

(i)
$$\begin{aligned}\sqrt{8} &= \sqrt{2 \times 2 \times 2}\\ &= \sqrt{2} \times \sqrt{2} \times \sqrt{2}\\ &= \left(\sqrt{2}\right)^2 \times \sqrt{2}\\ &= 2\sqrt{2}\end{aligned}$$

(ii)
$$\begin{aligned}\sqrt{6} \times \sqrt{3} &= \sqrt{6 \times 3}\\ &= \sqrt{2 \times 3 \times 3}\\ &= \left(\sqrt{3}\right)^2 \times \sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

(iii) $\sqrt{32} - \sqrt{18} = \sqrt{16 \times 2} - \sqrt{9 \times 2}$

$= 4\sqrt{2} - 3\sqrt{2}$

$= \sqrt{2}$

Look for square factors of 32 and 18.

16 is the largest square factor of 32.

9 is the largest square factor of 18.

(iv) $(4 + \sqrt{3})(4 - \sqrt{3}) = 16 - 4\sqrt{3} + 4\sqrt{3} - (\sqrt{3})^2$

$= 16 - 3$

$= 13$

Discussion point

→ What is a rational number?

Notice that in part (iv) of Example 1.28 there is no square root in the answer. In the next example, all the numbers involve fractions with a square root as part of the denominator. It is easier to work with numbers if any square roots are only part of the numerator. Manipulating a number to that form is called *rationalising the denominator.*

When the numerator and the denominator of a fraction are multiplied by the same number, then the value of the fraction stays the same. For example, $\frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10}$. We use this principle when rationalising a denominator. In the next examples, and question 3 of Exercise 1G, the denominators have only one term. In each case, multiply both the numerator and denominator by this number, and then simplify.

Example 1.29

Simplify the following by rationalising their denominators.

(i) $\frac{2}{\sqrt{3}}$ (ii) $\sqrt{\frac{3}{5}}$ (iii) $\sqrt{\frac{3}{8}}$

Solution

(i) $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{2\sqrt{3}}{(\sqrt{3})^2}$

$= \frac{2\sqrt{3}}{3}$

(ii) $\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}}$

$= \frac{\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{\sqrt{3} \times \sqrt{5}}{(\sqrt{5})^2}$

$= \frac{\sqrt{15}}{5}$

(iii) $\sqrt{\frac{3}{8}} = \frac{\sqrt{3}}{\sqrt{8}}$

$= \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{3} \times \sqrt{2}}{2(\sqrt{2})^2}$

$= \frac{\sqrt{6}}{4}$

Exercise 1G

Do not use a calculator for this exercise.

① Simplify the following.

(i) $\sqrt{32}$ (ii) $\sqrt{125}$

(iii) $\sqrt{5} \times \sqrt{15}$ (iv) $\sqrt{8} - \sqrt{2}$

(v) $3\sqrt{27} - 6\sqrt{3}$ (vi) $4(3 + \sqrt{2}) - 3(5 - \sqrt{2})$

(vii) $4\sqrt{32} - 3\sqrt{8}$ (viii) $5(6 - \sqrt{3}) + 2(3 + 4\sqrt{3})$

(ix) $2\sqrt{125} + 6\sqrt{5}$ (x) $3\left(2\sqrt{2} - 3\sqrt{3}\right) - 2\left(3\sqrt{2} - 5\sqrt{3}\right)$

② Simplify the following.

(i) $(\sqrt{2} - 1)^2$ (ii) $(4 - \sqrt{5})(2 + \sqrt{5})$

(iii) $(2 - \sqrt{7})(\sqrt{7} - 1)$ (iv) $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

(v) $(3 + \sqrt{2})(5 - 2\sqrt{2})$ (vi) $(\sqrt{7} - 3)(2\sqrt{7} + 3)$

(vii) $(3\sqrt{3} - 2)(2\sqrt{3} - 3)$ (viii) $(\sqrt{5} - \sqrt{3})^2$

(ix) $(5 - 3\sqrt{2})(2\sqrt{2} - 1)$ (x) $(2\sqrt{2} + 3)^2$

③ Simplify the following by rationalising their denominators.

(i) $\dfrac{1}{\sqrt{3}}$ (ii) $\dfrac{5}{\sqrt{5}}$ (iii) $\dfrac{8}{\sqrt{6}}$

(iv) $\sqrt{\dfrac{2}{3}}$ (v) $\dfrac{2\sqrt{2}}{\sqrt{8}}$ (vi) $\sqrt{\dfrac{3}{7}}$

(vii) $\dfrac{21}{\sqrt{7}}$ (viii) $\dfrac{5}{3\sqrt{5}}$ (ix) $\dfrac{\sqrt{75}}{\sqrt{125}}$

(x) $\dfrac{8}{\sqrt{128}}$

④ Simplify the following by writing them as single fractions.

(i) $\dfrac{2}{3 - \sqrt{2}} + \dfrac{2}{3 + \sqrt{2}}$ (ii) $\dfrac{5}{2 - \sqrt{3}} - \dfrac{3}{2 + \sqrt{3}}$

(iii) $\dfrac{1}{5 - 2\sqrt{6}} + \dfrac{3}{5 + 2\sqrt{6}}$ (iv) $\dfrac{4}{4 + \sqrt{3}} - \dfrac{1}{4 - \sqrt{3}}$

⑤ (i) Use the expansion of $(a + b)^3$ to simplify $\left(3 + \sqrt{2}\right)^3$.

(ii) Use the expansion of $(a + b)^3$ to simplify $\left(2 + \sqrt{5}\right)^3$.

(iii) Use the expansion of $(a + b)^4$ to simplify $\left(2 - \sqrt{3}\right)^4$.

(iv) Use the expansion of $(a + b)^4$ to simplify $\left(1 + \sqrt{6}\right)^4$.

(v) Use the expansion of $(a + b)^5$ to simplify $\left(1 + \sqrt{5}\right)^5$.

(vi) Use the expansion of $(a + b)^5$ to simplify $\left(2 - \sqrt{5}\right)^5$.

⑥ Solve the following equations.

(i) $\sqrt{32} - v\sqrt{2} = \sqrt{8}$ (ii) $w\sqrt{18} + \sqrt{8} = \sqrt{98}$

(iii) $3\sqrt{3} + y\sqrt{12} = 2\sqrt{27}$ (iv) $x\sqrt{50} + \sqrt{18} = 5x\sqrt{8}$

⑦ Simplify $\left(2+\sqrt{3}\right)^6$.

⑧ Solve the following equations.

(i) $\dfrac{m}{\sqrt{3}}+\dfrac{1}{\sqrt{12}}=\sqrt{3}$

(ii) $\dfrac{3n}{\sqrt{2}}-\dfrac{n+4}{\sqrt{8}}=\sqrt{18}$

(iii) $\dfrac{2x}{\sqrt{5}}=\sqrt{20}+\dfrac{x}{\sqrt{45}}$

(iv) $3\left(\dfrac{x}{\sqrt{2}}+\sqrt{8}\right)=\dfrac{5x}{\sqrt{18}}+\dfrac{3x}{\sqrt{32}}$

⑨ Solve $\left(x-\sqrt{2}\right)^3=x^2\left(x-\sqrt{18}\right)+3\sqrt{8}$.

⑩ The area of a square is $7+4\sqrt{3}$.

Given that the length of each side of the square is of the form $m+n\sqrt{3}$ (where m and n are integers), work out the perimeter.

Leave your answer in the form $p+\sqrt{q}$, where p and q are integers.

Rationalising denominators with two terms

For a denominator with two terms, the multiplier we use is the denominator with one of its signs changed.

Example 1.30

Rationalise the denominator of

$$\frac{3\sqrt{2}}{4-\sqrt{5}}.$$

Solution

$$\begin{aligned}\frac{3\sqrt{2}}{4-\sqrt{5}} &= \frac{3\sqrt{2}}{4-\sqrt{5}}\times\frac{4+\sqrt{5}}{4+\sqrt{5}}\\ &= \frac{12\sqrt{2}+3\sqrt{2}\sqrt{5}}{16+4\sqrt{5}-4\sqrt{5}-(\sqrt{5})^2}\\ &= \frac{12\sqrt{2}+3\sqrt{10}}{16-5}\\ &= \frac{12\sqrt{2}+3\sqrt{10}}{11}\end{aligned}$$

Example 1.31

Write $\dfrac{2\sqrt{3}-4}{3\sqrt{3}+5}$ in the form $a+b\sqrt{3}$, where a and b are integers.

Solution

$$\begin{aligned}\frac{2\sqrt{3}-4}{3\sqrt{3}+5} &= \frac{2\sqrt{3}-4}{3\sqrt{3}+5}\times\frac{3\sqrt{3}-5}{3\sqrt{3}-5}\\ &= \frac{6(\sqrt{3})^2-10\sqrt{3}-12\sqrt{3}+20}{9(\sqrt{3})^2-15\sqrt{3}+15\sqrt{3}-25}\\ &= \frac{18-22\sqrt{3}+20}{27-25}\\ &= \frac{38-22\sqrt{3}}{2}\\ &= 19-11\sqrt{3}\end{aligned}$$

Exercise 1H

Do not use a calculator for this exercise.

① Rationalise the denominators of these fractions:

(i) $\dfrac{2\sqrt{3}}{5+\sqrt{2}}$ (ii) $\dfrac{\sqrt{7}}{4-\sqrt{2}}$ (iii) $\dfrac{3\sqrt{3}}{\sqrt{3}+1}$

(iv) $\dfrac{2+\sqrt{2}}{3-\sqrt{2}}$ (v) $\dfrac{\sqrt{7}-3}{1-\sqrt{7}}$ (vi) $\dfrac{10+\sqrt{3}}{\sqrt{3}+\sqrt{2}}$.

② Write $\dfrac{3\sqrt{2}+6}{\sqrt{2}-1}$ in the form $a+b\sqrt{2}$, where a and b are integers.

③ Write $\dfrac{2\sqrt{5}}{4\sqrt{5}+9}$ in the form $c\sqrt{5}+d$, where c and d are integers.

④ Write $\dfrac{1+\sqrt{3}}{3+2\sqrt{3}}$ in the form $p+\frac{q}{r}\sqrt{3}$, where p, q and r are integers.

⑤ A rectangle has a width of $2+\sqrt{5}$ and an area of $1+\sqrt{5}$.
Work out its length.

⑥ Simplify $\dfrac{19}{\sqrt{27}-\sqrt{8}}$.

⑦ A triangle has an area of $11\sqrt{2}-2$ and a base length of $2+\sqrt{18}$.
Work out its perpendicular height.

⑧ The area of a trapezium is $4+\sqrt{27}$.
The lengths of its parallel sides are $3+\sqrt{12}$ and $2-\sqrt{3}$.
Work out the perpendicular distance between the parallel sides.

8 The product rule for counting

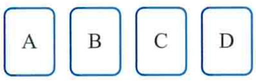


Figure 1.7

The product rule for counting is an efficient method for finding the number of combinations given a particular condition.

Consider the four cards in Figure 1.7.

In how many different ways can the four cards be arranged?

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

As there are only 24 arrangements it is possible to list them all and then count them.

However, this strategy is impractical if the number of arrangements is large.

Instead, consider the number of possibilities for each position. There are 4 letters which could be chosen first. This leaves 3 letters for the second position. So there are $4 \times 3 = 12$ ways of choosing the first two letters. For each of these, there are 2 letters to choose for the third position, leaving just 1 for the final position.

A quick way to calculate the number of arrangements is $4 \times 3 \times 2 \times 1 = 24$.

Example 1.32

Here are seven cards.

2	3	4	5	6	7	8

Figure 1.8

Using all seven cards, in how many different ways can they be arranged to form a seven-digit number?

Solution

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Calculations such as this are sometimes made easier using the factorial function on a calculator. The factorial function is defined as

$$n! = \begin{cases} 1 & n = 0 \\ n \times (n-1)! & n \in \mathbb{N} \end{cases}$$

$\mathbb{N}$ is the set of natural numbers, i.e. positive integers.

For example,

$$\begin{aligned} 4! &= 4 \times 3! \\ &= 4 \times 3 \times 2! \\ &= 4 \times 3 \times 2 \times 1! \\ &= 4 \times 3 \times 2 \times 1 \times 0! \\ &= 4 \times 3 \times 2 \times 1 \times 1 \\ &= 24 \end{aligned}$$

Note: $n!$ is more simply remembered as the product of all the natural numbers up to and including n.

ACTIVITY 1.3

Find the factorial function on your calculator. Try different inputs.

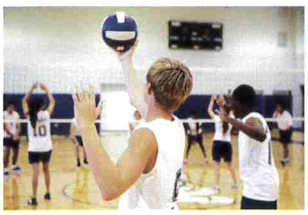
For which numbers does the factorial function not work?

Try $-5!$

Then try $(-5)!$

Why does the calculator respond differently?

Example 1.33



In a volleyball team, six players start in six different positions.

At the start of a game, how many different arrangements are possible for the six players?

Solution

$$\begin{aligned} 6 \times 5 \times 4 \times 3 \times 2 \times 1 &= 6! \\ &= 720 \end{aligned}$$

Note

This specification does not assess a candidate's knowledge of the factorial function.

FUTURE USES

At A-Level, the factorial function is used to extend this topic further. It is also used in algebraic expansions.

Example 1.34

A volleyball team of six players is chosen from a squad of twelve.

(i) How many different starting arrangements are possible?

(ii) How many different teams can be selected?

Solution

(i) There are 12 possibilities for the first position, 11 for the second position, 10 for the third, then 9, and so on.

$12 \times 11 \times 10 \times 9 \times 8 \times 7 = 665\,280$

(ii) As you saw in Example 1.33, each team can be arranged in 720 different ways. In this second part of the question, the arrangement of each team is not important. So the number of different teams is

$665\,280 \div 720 = 924$

Discussion point

→ Is it possible to use the factorial function to make this calculation easier?

Example 1.35

Using each of the following digits no more than once, 3 1 7 9 8 5

(i) how many different four-digit numbers can be made?

(ii) how many **even** three-digit numbers can be made?

(iii) how many numbers less than 4000 can be made?

(iv) how many **odd** numbers greater than 500 000 can be made?

Solution

(i) $6 \times 5 \times 4 \times 3 = 360$

(ii) An even three-digit number will have an even number as the last digit. Hence, 8 must be the last digit.

So there are five possibilities for the first digit, leaving four for the second digit.

$1 \times 5 \times 4 = 20$

(iii) This question does not specify the number of digits to be used, so each must be considered separately.

Only 6 one-digit numbers are possible.

There are $6 \times 5 = 30$ two-digit numbers,

and $6 \times 5 \times 4 = 120$ three-digit numbers.

When considering the four-digit numbers, the first digit must be either a 1 or a 3. So the number of possible four-digit numbers is $2 \times 5 \times 4 \times 3 = 120$

This gives a total of $6 + 30 + 120 + 120 = 276$

(iv) As the number must be greater than 500 000 then the first digit must be a 5, or a 7, or an 8, or a 9.

If the first digit is a 5, or a 7, or a 9, then there are only four possibilities for the last digit. If 8 is the first digit, then there are five possible last digits.

5____1	5____3	5____7	5____9	
7____1	7____3	7____5	7____9	
8____1	8____3	8____5	8____7	8____9
9____1	9____3	9____5	9____7	

So there are 17 possibilities for the first and last digits.

The remaining four digits have $4 \times 3 \times 2 \times 1 = 24$ arrangements.

So there are $17 \times 24 = 408$ odd numbers greater than 500 000 which use the above digits only once.

Example 1.36

In 2001 a new system for vehicle registration plates was introduced.

Each plate starts with two letters (excluding I, Q, Z) which identify the area in which the vehicle was registered.

These are followed by two digits (except 01) which identify the age of the vehicle.

Finally, there are three more letters, excluding I and Q.

How many such registration plates are possible?

23 letters, excluding I, Q, Z

99 two-digit numbers, excluding 01

24 letters, excluding I and Q

Solution

$23 \times 23 \times 99 \times 24 \times 24 \times 24 = 723\,976\,704$

Exercise 1I

① Work out the number of arrangements of the letters ABCDE.

② Matt, Joe, Ben, Saul, Chris, Anna and Steve stand in a straight line.

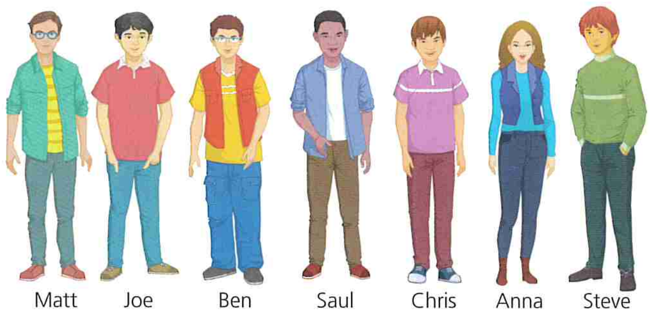


Figure 1.9

How many arrangements are possible?

③ Here are seven digits: 9 5 1 4 2 3 8

(i) If each digit can appear no more than once, how many six-digit numbers can be formed?

(ii) If each of the six digits can appear more than once, how many six-digit numbers can be formed?

④ How many **even** numbers can be made using each of the digits 2, 3, 7, 8 exactly once?

⑤ Work out the number of three-digit multiples of 5

⑥ Ten books stand next to each other on a shelf.

How many different arrangements are possible?

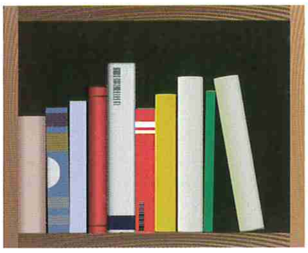


Figure 1.10

PS ⑦ A toy box has six compartments. Four different toys are to be put in the box.

(i) If each compartment can hold only one toy, how many arrangements are possible?

(ii) If each compartment can hold up to four toys, how many arrangements are possible?

PS ⑧ Figure 1.11 shows the net of a cube.

The numbers 1, 2, 3, 4, 5, 6 are to be written on the net – one number per square.

(i) How many different ways can the numbers be written on the net?

(ii) If the net is folded into a cube, the numbers on opposite faces must add to 7

In this case, how many ways can the numbers be written on the net?

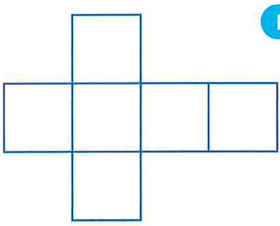


Figure 1.11

PS ⑨ A palindromic integer is a whole number which reads the same forwards and backwards.

Two examples of palindromic integers are 15 751 and 302 203

This specification will not assess candidates' knowledge of palindromes.

(i) Work out the number of three-digit palindromic integers.

(ii) Work out the number of four-digit palindromic integers.

(iii) How many palindromic integers are less than one million?

(iv) How many six-digit palindromic integers are multiples of 9?

PS ⑩ Figure 1.12 shows a network comprising five points and the links between them.

A tour starts at one point, visits each of the other points, and then returns to its starting point.

For example, ACEDBA is a tour.

Work out the number of different tours.

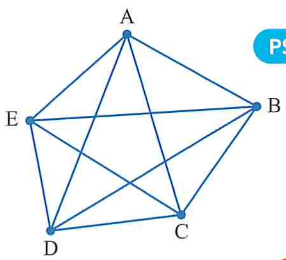


Figure 1.12

This specification will not assess candidates' knowledge of networks and tours.

PS ⑪ A network comprises eight points, with a direct link between each pair of points.

Work out the number of different tours.

(See question 10 for a description of a tour.)

PS ⑫ A five-digit number greater than 60 000 is to be made from these six cards.

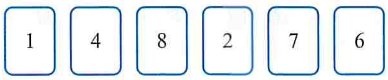


Figure 1.13

Each card can be used only once.

(i) How many five-digit numbers greater than 60 000 are possible?

(ii) How many of these numbers are even?

PS 13 A network comprises n points with a direct link between each pair of points. Work out the number of different tours in terms of n.

(See question 10 for a description of a tour. Note: this would not be assessed in the Level 2 Further Maths exam.)

PS 14 (i) How many different ways can a team of 5 people be arranged.

(ii) How many different teams of 5 can be selected from a group of 13 people? (Note: the arrangement of each team is not important.)

REAL-WORLD CONTEXT

This topic is developed further in A-Level mathematics and is often referred to as combinatorics. It has many applications in the real world, not least in password security.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- simplify an algebraic expression
- solve a linear equation
- solve percentage problems
- solve ratio problems
- work out the product of two (or more) algebraic expressions
- work out the expansion of an expression of the form $(a+b)^n$, where n is a positive integer
- manipulate expressions involving surds, including
  - simplifying surds
  - adding/subtracting compatible surds
  - rationalising a denominator of the form $\sqrt{a}$
  - rationalising a denominator of the form $a+\sqrt{b}$
  - rationalising a denominator of the form $\sqrt{a}+\sqrt{b}$
- efficiently find the number of combinations given a particular condition.

KEY POINTS

1 Simplify algebraic expressions by collecting like terms and/or expanding brackets.

2 Add/subtract algebraic expressions by rewriting with common denominators.

3 Simplify fractions by cancelling common factors in the numerator and denominator.

4 An expansion of $(a+b)^n$ comprises $n+1$ terms, each of the form Pa^qb^r where P is a number from Pascal's triangle, and $q+r=n$.

5 When simplifying surds

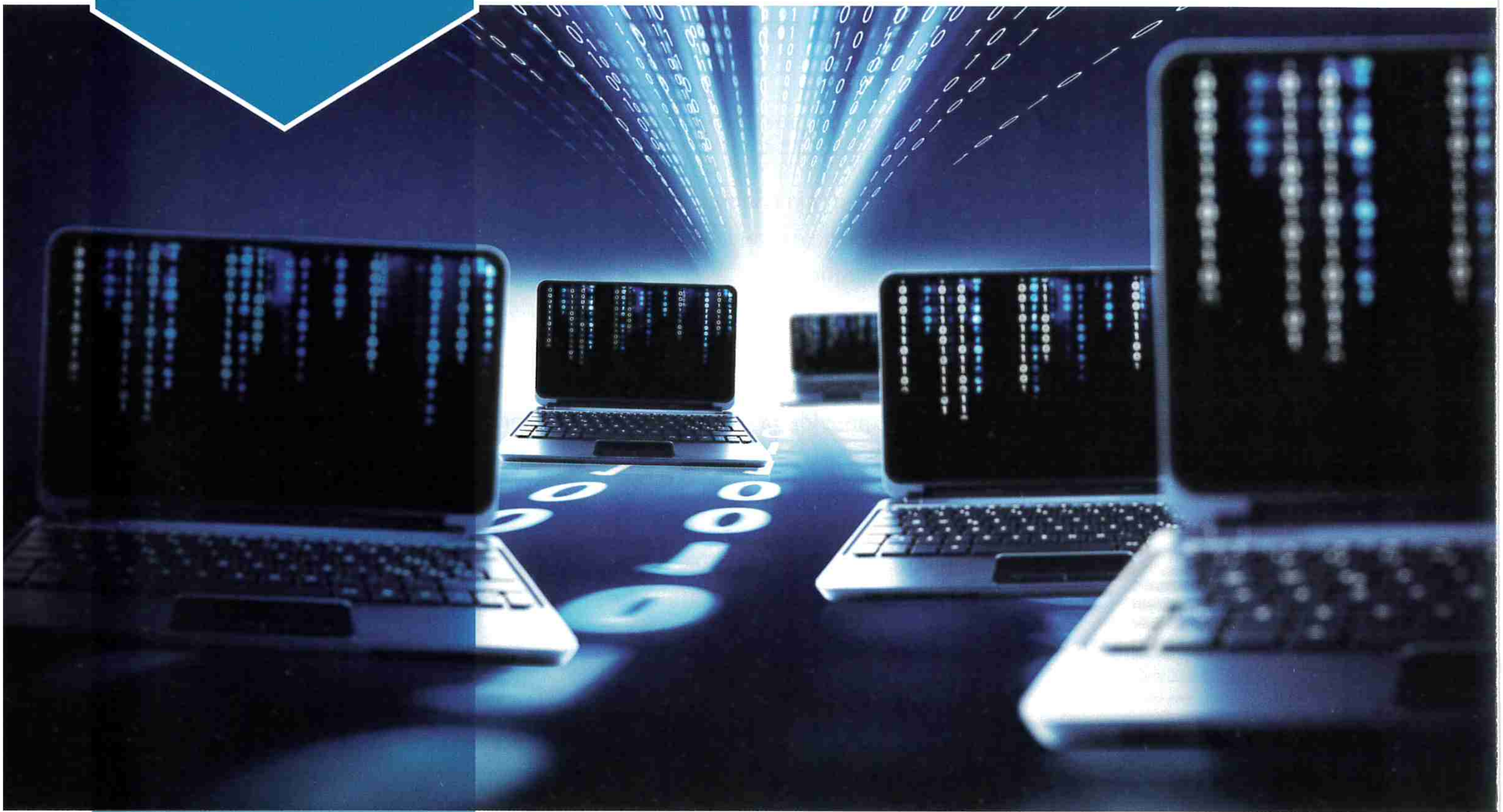
- only like surds can be added or subtracted,
- $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$.

6 A denominator can be rationalised as follows:

- $\sqrt{a}$ can be rationalised by using the multiplier $\sqrt{a}$,
- $a+\sqrt{b}$ can be rationalised by using the multiplier $a-\sqrt{b}$,
- a two-term expression can be rationalised using a multiplier found by changing one sign.

7 When applying the product rule of counting, consider the number of options as each item is selected, and multiply them to find the number of possibilities.

2 Algebra II



If A equals success, then the formula is A equals x plus y plus z, with x being work, y play, and z keeping your mouth shut.
Albert Einstein

Prior knowledge

Students are expected to be able to identify a common factor of two or more terms. This may be just a number, a letter or both; see Chapter 1.

1 Factorising

Factorising an expression involves writing the expression as a product using brackets. Simple cases of this were seen in Chapter 1 Number and algebra 1. Here, pairs of brackets will be needed. If you have already learnt another method, and use it quickly and accurately, then you should stick with it. With practice, you may be able to factorise some of these expressions **by inspection**.

You will meet factorising again in Chapter 4.

! When factorising a quadratic with no constant term, only one bracket is required. For example, $2x^2 - 8x = 2x(x - 4)$.

Example 2.1

Factorise $xa + xb + ya + yb$.

Solution

First take out a common factor of each pair of terms.

$$\Rightarrow \quad xa + xb + ya + yb = x(a + b) + y(a + b)$$

Next notice that $(a + b)$ is now a common factor.

$$\Rightarrow \quad x(a + b) + y(a + b) = (a + b)(x + y)$$

In practice this can relate to areas of rectangles.

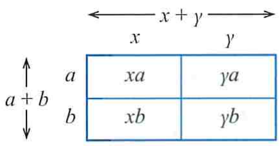


Figure 2.1

The idea illustrated in Figure 2.1 can be used to factorise a quadratic expression containing three terms, but first you must decide how to split up the term in x.

Example 2.2

Factorise $x^2 + 6x + 8$

Solution

Splitting the $6x$ as $4x + 2x$ gives

> Why would you choose to split up $6x$ this way?

$$x^2 + 6x + 8 = x^2 + 4x + 2x + 8$$
$$= x(x + 4) + 2(x + 4)$$
$$= (x + 4)(x + 2)$$

> **Discussion point**
> → Is the illustration in Figure 2.2 the only possibility?

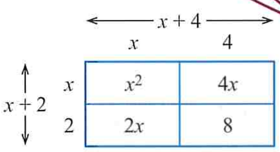


Figure 2.2

The crucial step is knowing how to split up the middle term.

To answer this question, notice that

- the numbers 4 and 2 have a sum of 6, which is the **coefficient** of x (i.e. the number multiplying x) in $x^2 + 6x + 8$
- the numbers 4 and 2 have a product of 8 which is the **constant term** in $x^2 + 6x + 8$

There is only one pair of numbers that satisfies both of these conditions.

Example 2.3

Factorise $x^2 - 7x - 18$

Solution

Pairs of numbers with a product of (-18) are:

1 and (-18)

2 and (-9)

3 and (-6)

6 and (-3)

9 and (-2)

18 and (-1)

> Since the pair of numbers that you are looking for is unique, you can stop listing products when you find one that has the correct sum.

> **Discussion point**
> → Do you get the same factors if the order in which you use the 2 and the (-9) is reversed so that you write it $x^2 - 9x + 2x - 18$?

There is only one pair, 2 and (−9), with a sum of (−7) so use these.

$$\begin{aligned} x^2 - 7x - 18 &= x^2 + 2x - 9x - 18 \\ &= x(x + 2) - 9(x + 2) \\ &= (x + 2)(x - 9) \end{aligned}$$

Notice the sign change due to the – sign in front of the 9.

Example 2.4

Factorise $x^2 - 16$

Solution

First write $x^2 - 16 = x^2 + 0x - 16$

Pairs of numbers with a product of (−16) are:

1 and (−16)

2 and (−8)

4 and (−4),...(stop here)

The only pair with a sum of 0 is 4 and (−4).

$$\begin{aligned} x^2 - 16 &= x^2 + 4x - 4x - 16 \\ &= x(x + 4) - 4(x + 4) \\ &= (x + 4)(x - 4) \end{aligned}$$

This is an example of a special case called **the difference of two squares** since you have $x^2 - 4^2 = (x + 4)(x - 4)$.

In general, $a^2 - b^2 = (a + b)(a - b)$.

Most people recognise this when it occurs and write down the answer straight away.

Example 2.5

Factorise $4x^2 - 9y^2$.

Solution

$$\begin{aligned} 4x^2 - 9y^2 &= (2x)^2 - (3y)^2 \\ &= (2x + 3y)(2x - 3y) \end{aligned}$$

Notice that this technique can be extended to any situation where the coefficients of the two terms are square numbers.

Example 2.6

Factorise fully $y^5 - 36y^3$.

Solution

The instruction to factorise fully tells you that there is likely to be more than one step involved.

Take out the highest common factor of the two terms $y^5 - 36y^3 = y^3(y^2 - 36)$.

Use the difference of two squares $y^3(y^2 - 36) = y^3(y + 6)(y - 6)$.

The technique for finding how to split the middle term needs modifying for examples where the expression starts with a multiple of x^2. The difference is that you now multiply the two outside numbers together to give the product you want.

Example 2.7

Factorise $2x^2 - 11x + 15$

Solution

A negative sum and a positive product means that both numbers are negative.

Here the sum is (−11) and the product is $2 \times 15 = 30$

Options to give the correct product are:

(−1) and (−30) (−5) and (−6)

(−2) and (−15)

(−3) and (−10)

(−5) and (−6), in either order, are the only options giving a sum of (−11).

$$\begin{aligned} 2x^2 - 11x + 15 &= 2x^2 - 5x - 6x + 15 \\ &= x(2x - 5) - 3(2x - 5) \\ &= (x - 3)(2x - 5) \end{aligned}$$

Discussion point

→ Try this example writing $2x^2 - 11x + 15$ as $2x^2 - 6x - 5x + 15$

Example 2.8

Factorise $3x^2 - 10xy - 8y^2$.

Solution

This expression can be factorised using the same method used in the previous example.

Here the sum is (−10) and the product is $3 \times -8 = -24$

A negative product means that one number is positive and the other is negative.

Option needed is (−12) and 2

$$\begin{aligned} 3x^2 - 10xy - 8y^2 &= 3x^2 - 12xy + 2xy - 8y^2 \\ &= 3x(x - 4y) + 2y(x - 4y) \\ &= (3x + 2y)(x - 4y) \end{aligned}$$

Example 2.9

Factorise $(x + 3)^2 - 4y^2$.

Solution

This example uses the difference of two squares.

Writing the expression as $(x + 3)^2 - (2y)^2$ and factorising gives

$$[(x + 3) + (2y)]\,[(x + 3) - 2y]$$

$$= (x + 3 + 2y)(x + 3 - 2y)$$

Exercise 2A

① Factorise the following expressions.

(i) $ab - ac + db - dc$ (ii) $2xy + 2x + wy + w$
(iii) $2pq - 8p - 3rq + 12r$ (iv) $5 - 5m - 2n + 2nm$

② Factorise the following expressions.

(i) $x^2 + 5x + 6$ (ii) $y^2 - 5y + 4$ (iii) $m^2 - 8m + 16$
(iv) $m^2 - 8m + 15$ (v) $x^2 + 3x - 10$ (vi) $a^2 + 20a + 96$
(vii) $x^2 - x - 6$ (viii) $y^2 - 16y + 48$ (ix) $k^2 + 10k + 24$
(x) $k^2 - 10k - 24$

③ Each of these is a difference of two squares. Factorise them.

(i) $x^2 - 4$ (ii) $a^2 - 25$ (iii) $9 - p^2$
(iv) $x^2 - y^2$ (v) $t^2 - 64$ (vi) $4x^2 - 1$
(vii) $4x^2 - 9$ (viii) $4x^2 - y^2$ (ix) $16x^2 - 25$
(x) $9a^2 - 4b^2$

④ Factorise the following expressions.

(i) $2x^2 + 5x + 2$ (ii) $2a^2 + 11a - 21$ (iii) $15p^2 + 2p - 1$
(iv) $3x^2 + 8x - 3$ (v) $5a^2 - 9a - 2$ (vi) $2p^2 + 5p - 3$
(vii) $8x^2 + 10x - 3$ (viii) $2a^2 - 3a - 27$ (ix) $9x^2 - 30x + 25$
(x) $4x^2 + 4x - 15$

Example 2.8 shows you how to factorise an expression where there are two variables.

⑤ Factorise the following expressions.

(i) $x^2 + 3xy + 2y^2$ (ii) $x^2 + 4xy - 5y^2$ (iii) $a^2 - ab - 12b^2$
(iv) $c^2 - 11cd + 24d^2$ (v) $x^2 + 9xy + 20y^2$ (vi) $p^2 + 2pr - 15r^2$
(vii) $a^2 - 2ar - 15r^2$ (viii) $s^2 - 4st + 4t^2$ (ix) $m^2 - 5mn - 6n^2$
(x) $r^2 + 2rs - 8s^2$

⑥ Factorise the following expressions. (This question extends factorising using the difference of two squares as in Example 2.9.)

(i) $(2a + 1)^2 - a^2$ (ii) $(3x + 1)^2 - (x + 4)^2$
(iii) $(2p - 3)^2 - (p + 1)^2$ (iv) $16 - (5y - 2)^2$
(v) $(2a + 1)^2 - a^2$ (vi) $(3x + 1)^2 - (x + 4)^2$
(vii) $(2p - 3)^2 - (p + 1)^2$ (viii) $9 - (2y - 3)^2$

⑦ Factorise the following expressions.

(i) $2x^2 + 5xy + 2y^2$ (ii) $3x^2 + 5xy - 2y^2$
(iii) $5a^2 - 8ab + 3b^2$ (iv) $6c^2 + 5cd - 4d^2$
(v) $6p^2 - 37pq + 6q^2$ (vi) $7g^2 + 5gh - 2h^2$
(vii) $6h^2 - 5hk - 4k^2$ (viii) $8w^2 - 6wx + x^2$

⑧ Factorise fully the following expressions.

(i) $x^3 - 4x$ (ii) $a^4 - 16a^2$
(iii) $9y^3 - y^5$ (iv) $2x^3 - 2x$
(v) $4p^4 - 9p^2$ (vi) $100x - x^3$
(vii) $18c^3 - 2c$ (viii) $8x^3 - 50xy^2$

ACTIVITY 2.1

(i) Work out 9^2 and $(a^2)^2$. ← Remember that $(a^p)^q = a^{pq}$.
(ii) Show that $a^4 - 81$ is the difference of two squares.
(iii) Factorise fully $a^4 - 81$

ACTIVITY 2.2

(i) Factorise $10x^2 + 11x + 3$

(ii) Factorise $10(p+q)^2 + 11(p+q) + 3$

2 Rearranging formulae

C is called the **subject** of the formula.

The circumference of a circle is given by $C = 2\pi r$ where r is the radius. An equation such as this is often called a formula.

In some cases, you want to calculate r directly from C. You want r to be the subject of the formula.

Example 2.10

Make r the subject of $C = 2\pi r$.

! Notice how the new subject should be on its own on the left-hand side of the new formula and must not appear on the right-hand side.

Solution

Divide both sides by $2\pi \Rightarrow \dfrac{C}{2\pi} = r$

$\Rightarrow r = \dfrac{C}{2\pi}$

Example 2.11

Make x the subject of this formula.

$h = \sqrt{(x^2 + y^2)}$

A square root is assumed to be positive unless $\pm$ is added in front of it.

Discussion point

→ What would you do with the ± sign in the case where h is the hypotenuse of a right-angled triangle with x and y as the other two sides?

Solution

Square both sides $\Rightarrow h^2 = x^2 + y^2$

Subtract y^2 from both sides $\Rightarrow h^2 - y^2 = x^2$

Make the x^2 term the subject $\Rightarrow x^2 = h^2 - y^2$

Take the square root of both sides $\Rightarrow x = \pm\sqrt{(h^2 - y^2)}$

Example 2.12

Make a the subject of this formula.

$v = u + at$

Solution

Subtract u from both sides $\Rightarrow v - u = at$

Divide both sides by t $\Rightarrow \dfrac{v-u}{t} = a$

Write the answer with a on the left-hand side $\Rightarrow a = \dfrac{v-u}{t}$

Exercise 2B

In this exercise all the equations refer to formulae used in mathematics. How many of them do you recognise?

① Make

(i) u (ii) t

the subject of $v = u + at$.

② Make b the subject of $A = \frac{1}{2}bh$.

③ Make l the subject of $P = 2(l + b)$.

④ Make r the subject of $A = \pi r^2$.

⑤ Make c the subject of $A = \frac{1}{2}(b + c)h$.

⑥ Make h the subject of $A = \pi r^2 + 2\pi rh$.

⑦ Make l the subject of $T = \frac{\lambda e}{l}$.

⑧ Make

(i) u (ii) a

the subject of $s = ut + \frac{1}{2}at^2$.

⑨ Make x the subject of $v^2 = \omega^2(a^2 - x^2)$.

The following examples show how to rearrange a formula when the letter that is to be the subject appears more than once.

Example 2.13

Make t the subject of this formula.

$at = 3(t + 2)$

Solution

Expand the brackets	$\Rightarrow$	$at = 3t + 6$
Collect all the terms in t on one side	$\Rightarrow$	$at - 3t = 6$
Factorise	$\Rightarrow$	$t(a - 3) = 6$
Divide both sides by $(a - 3)$	$\Rightarrow$	$t = \frac{6}{a - 3}$

The brackets are not needed in the denominator.

Example 2.14

Make x the subject of this formula.

$$y = \frac{x + 2}{1 + 3x}$$

Solution

Multiply both sides by $(1 + 3x)$	$\Rightarrow$	$y(1 + 3x) = x + 2$
Expand the brackets	$\Rightarrow$	$y + 3xy = x + 2$
Collect all the terms in x on one side and all the other terms on the other side	$\Rightarrow$	$3xy - x = 2 - y$
Factorise	$\Rightarrow$	$x(3y - 1) = 2 - y$
Divide both sides by $(3y - 1)$	$\Rightarrow$	$x = \frac{2 - y}{3y - 1}$

Exercise 2C

1. Make m the subject of $3m = x(m + 2)$.
2. Make y the subject of $5y - 2x = xy$.
3. Make b the subject of $4(a + b) = 3(a - b)$.
4. Make h the subject of $S = 2\pi r^2 + 2\pi rh$.
5. Make x the subject of $y = \dfrac{x + 1}{2 + x}$.
6. Make c the subject of $d(2 + c) = 1 - 3c$.
7. (i) Make t the subject of $x = \dfrac{t}{t - 3}$.
   (ii) Hence, or otherwise, work out the value of t when $x = 3$
8. (i) Make p the subject of $r = \dfrac{3p + 2}{2p + 3}$.
   (ii) Hence, or otherwise, work out the value of p when $r = -1$

ACTIVITY 2.3

(i) (a) Show that $(x + 3)^2 = x^2 + 6x + 9$
(b) Hence make x the subject of $y = x^2 + 6x + 9$
(ii) (a) Show that $(x - 5)^2 + 4 = x^2 - 10x + 29$
(b) Hence make x the subject of $p = x^2 - 10x + 29$

3 Simplifying algebraic fractions

Prior knowledge

Students should aim to be able to cancel fractions, find the least common multiple of two numbers and be able to identify a least common denominator of two or more fractions.

Discussion points

➜ What is a fraction in arithmetic?
➜ What about in algebra?

Fractions in algebra obey the same rules as fractions in arithmetic.

These cover two pairs of operations: × and ÷, and + and −.

Discussion points

➜ When can you cancel fractions in arithmetic?
➜ What about in algebra?
➜ What is a factor in arithmetic?
➜ What about in algebra?

Simplifying algebraic fractions

Example 2.15

Simplify the following.

(i) $\frac{18}{24}$

(ii) $\frac{2x+2}{3x+3}$

(iii) $\frac{a^2-a-6}{a^2-8a+15}$

Discussion points

→ Look at this calculation for (ii).

$\frac{2\cancel{x}+2}{3\cancel{x}+3}=\frac{4}{6}=\frac{2}{3}$

Why is it wrong?

→ Look at this calculation for (iii).

$\frac{{}^{1}\cancel{a^2}-\cancel{a}^{1}-6}{{}_{1}\cancel{a^2}-8\cancel{a}_{1}+15}$

$=-\frac{6}{8}=-\frac{3}{4}$

Why is it wrong?

Solution

(i) $\frac{18}{24}=\frac{{}^{1}\cancel{6}\times 3}{{}_{1}\cancel{6}\times 4}=\frac{3}{4}$

(ii) $\frac{2x+2}{3x+3}=\frac{2\cancel{(x+1)}^{1}}{3\cancel{(x+1)}_{1}}=\frac{2}{3}$

(iii) $\frac{a^2-a-6}{a^2-8a+15}=\frac{\cancel{(a-3)}^{1}(a+2)}{\cancel{(a-3)}_{1}(a-5)}=\frac{a+2}{a-5}$

Example 2.16

Simplify the following.

(i) $\frac{2}{3}\times\frac{9}{14}$

(ii) $\frac{3}{4}\div\frac{9}{16}$

(iii) $\frac{3a^2b}{2c}\times\frac{4c^3}{9ab}$

(iv) $\frac{4n^2-9}{n+1}\div\frac{2n+3}{n^2-1}$

Discussion point

→ Look at this calculation for (iv).

$\frac{\cancel{4n^2}^{2n}-\cancel{9}^{3}}{\cancel{n}_{1}+1}\times\frac{\cancel{n^2}^{n}-1}{\cancel{2n}_{1}+\cancel{3}_{1}}$

$=\frac{(2n-3)(n-1)}{4}$

Why is it wrong?

Solution

(i) $\frac{{}^{1}\cancel{2}}{{}_{1}\cancel{3}}\times\frac{\cancel{9}^{3}}{\cancel{14}_{7}}=\frac{1\times 3}{1\times 7}=\frac{3}{7}$

(ii) $\frac{3}{4}\div\frac{9}{16}=\frac{\cancel{3}^{1}}{\cancel{4}_{1}}\times\frac{\cancel{16}^{4}}{\cancel{9}_{3}}=\frac{4}{3}$

(iii) $\frac{{}^{1}\cancel{3}a^2\cancel{b}}{{}_{1}\cancel{2}\cancel{c}}\times\frac{{}^{2}\cancel{4}c^{\cancel{3}2}}{{}_{3}\cancel{9}\cancel{a}b}=\frac{2ac^2}{3}$

(iv) $\frac{4n^2-9}{n+1}\div\frac{2n+3}{n^2-1}=\frac{\cancel{(2n+3)}^{1}(2n-3)}{\cancel{(n+1)}_{1}}\times\frac{\cancel{(n+1)}^{1}(n-1)}{\cancel{(2n+3)}_{1}}$

$=(2n-3)(n-1)$

Example 2.17

Simplify the following.

(i) $\frac{2}{3}+\frac{3}{4}$

(ii) $\frac{5x}{6}+\frac{x}{4}$

(iii) $\frac{2}{(x+1)}+\frac{5}{(x-1)}$

(iv) $\frac{a}{a^2-1}-\frac{2}{a+1}$

Solution

(i) $\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$

(ii) $\frac{5x}{6} + \frac{x}{4} = \frac{10x}{12} + \frac{3x}{12} = \frac{13x}{12}$

Take care to ensure that the common denominator is the least common multiple of the original denominators.

(iii) $\frac{2}{(x+1)} + \frac{5}{(x-1)} = \frac{2(x-1)}{(x+1)(x-1)} + \frac{5(x+1)}{(x+1)(x-1)}$

$= \frac{2x-2+5x+5}{(x+1)(x-1)}$

$= \frac{7x+3}{(x+1)(x-1)}$

(iv) $\frac{a}{a^2-1} - \frac{2}{a+1} = \frac{a}{(a-1)(a+1)} - \frac{2}{a+1}$

$= \frac{a}{(a-1)(a+1)} - \frac{2(a-1)}{(a-1)(a+1)}$

$= \frac{a-2a+2}{(a-1)(a+1)}$

$= \frac{2-a}{(a-1)(a+1)}$

Exercise 2D

① Simplify the following.

(i) $\frac{2(x+3)}{4x+12}$ (ii) $\frac{4x-8}{(x-2)(x+8)}$ (iii) $\frac{3(x+y)}{x^2-y^2}$

(iv) $\frac{6x^2y^3}{9xy^4}$ (v) $\frac{2p}{6p-2p^2}$ (vi) $\frac{4ab^3}{10a^3b}$

② Simplify the following.

(i) $\frac{x^2-4x+3}{2x-6}$ (ii) $\frac{x^2+xy}{x^2-y^2}$ (iii) $\frac{a+2}{a^2-a-6}$

(iv) $\frac{3x^2+15x}{10x+2x^2}$ (v) $\frac{9x^2-1}{9x+3}$ (vi) $\frac{3x^2+3xy}{6xy+6y^2}$

③ Simplify the following.

(i) $\frac{3a}{b^2} \times \frac{b^3}{6a}$ (ii) $\frac{xy-y^2}{y} \times \frac{x}{x-y}$

(iii) $\frac{x^2-y^2}{y} \times \frac{x}{x-y}$ (iv) $\frac{x+1}{2x} \div \frac{4x^2-4}{x^2}$

(v) $\frac{3a^2+a-2}{2} \div \frac{6a^2-a-2}{8a+4}$ (vi) $\frac{2p^2-pq-q^2}{3p+3q} \div \frac{2p^2-3pq+q^2}{2p+2q}$

④ Simplify the following.

(i) $\frac{x^2-4x+4}{x^2-2x} \times \frac{x-2}{x^2-4}$ (ii) $\frac{2x-1}{x+1} \div \frac{2x^2-x-1}{x^2+3x+2}$

(iii) $\frac{4p^2+12}{p-3} \times \frac{p^2-9}{p^2+3}$ (iv) $\frac{3x^2-9}{x+2} \div \frac{x^2-6x+9}{x^2+x-2}$

(v) $\frac{3a^2-12}{5a^2-4a-1} \times \frac{5a+1}{(a-2)^2}$ (vi) $\frac{2t}{t^2+1} \div \frac{4t^2}{t^4-1}$

⑤ Simplify the following.

(i) $\frac{3a}{5} - \frac{a}{4}$ (ii) $\frac{5}{3a} - \frac{4}{a}$

(iii) $\frac{2}{(m+n)} - \frac{1}{(m-n)}$ (iv) $\frac{4}{p-2} - \frac{3}{2p+1}$

(v) $\frac{1}{2(x-1)} + \frac{2}{(x+4)}$ (vi) $\frac{1}{2(a-1)} + \frac{2}{3(a+4)}$

⑥ Simplify the following.

(i) $\frac{2}{a^2+a} + \frac{3}{a^2-a}$ (ii) $\frac{2x}{x-y} + \frac{2y}{y-x}$

(iii) $\frac{p}{p^2-1} - \frac{1}{p+1}$ (iv) $\frac{a-b}{a+b} + \frac{a+b}{a-b}$

(v) $\frac{4}{x^2-4} - \frac{3}{x+2}$ (vi) $\frac{7}{5(x-2)} - \frac{2}{x+4}$

⑦ Simplify the following.

(i) $\frac{1}{x+1} - \frac{2}{x+2} + \frac{3}{x+3}$ (ii) $\frac{3}{x+1} - \frac{2}{x-2} + \frac{4}{x+3}$

(iii) $\frac{x+2}{(x+1)^2} - \frac{1}{x}$

⑧ Simplify the following.

(i) $\frac{4t}{t^2+2t+1} + \frac{3}{t+1}$ (ii) $\frac{1}{y^2-x^2} + \frac{3}{y+x}$

(iii) $1 + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$

4 Solving linear equations involving fractions

Prior knowledge

Students are expected to be familiar with the basic rules for the mathematical operations +, −, × and ÷ which are used in Chapter 1.

Example 2.18

Solve the following.

$$\frac{x+2}{6} = \frac{x-6}{2}$$

Solution

The LCM of 6 and 2 is 6, so multiply by 6.

$$^1\cancel{6} \times \frac{(x+2)}{\cancel{6}_1} = {^3\cancel{6}} \times \frac{(x-6)}{\cancel{2}_1}$$

$$\Rightarrow x + 2 = 3x - 18$$

$$\Rightarrow 20 = 2x$$

$$\Rightarrow x = 10$$

Discussion point

➜ When you multiply a fraction by an integer, you only multiply its numerator (top line). Why?

Example 2.19

Solve the following.

$$\frac{x+2}{6} + 3 = \frac{x}{5}$$

Discussion point

→ Look at this version of the first stage of the solution for Example 2.19.

$30 \times \frac{(x+2)}{6} + 3$

$= 30 \times \frac{x}{5}$

Why is it wrong?

Solution

The LCM of 6 and 5 is 30, so multiply by 30.

$$\overset{5}{\cancel{30}} \times \frac{(x+2)}{\cancel{6}_1} + 30 \times 3 = \overset{6}{\cancel{30}} \times \frac{x}{\cancel{5}_1}$$

$\Rightarrow \quad 5x + 10 + 90 = 6x$

$\Rightarrow \quad x = 100$

Exercise 2E

Solve the following equations.

① $x - \frac{x}{5} = \frac{2}{3}$

② $\frac{2}{a} - \frac{3}{4a} = 2$

③ $\frac{x-4}{6} = \frac{x+2}{3}$

④ $\frac{2-3x}{6} = \frac{2}{3}$

⑤ $\frac{3p+2}{2} - \frac{p-1}{5} = 3$

⑥ $\frac{3(x-2)}{2} - \frac{x-5}{4} = 2$

⑦ $x + 1 - \frac{3(x-2)}{2} = 7$

⑧ $\frac{3(t+4)}{8} + 2 = \frac{2t}{3}$

5 Completing the square

When considering a quadratic expression it will sometimes be useful to write it to include the term $(x+a)^2$ or $(x-a)^2$, where a is a constant. Some uses of this approach will be seen later in sections on quadratic equations and quadratic graphs.

You will meet completing the square again in Chapter 4.

Example 2.20

Work out the values of p and q such that $x^2 - 6x + 2 = (x-p)^2 + q$.

Solution

Expand the bracket $x^2 - 6x + 2 = x^2 - 2px + p^2 + q$.

$$(x-p)^2 = (x-p)(x-p)$$
$$= x^2 - px - px + p^2$$
$$= x^2 - 2px + p^2$$

Equate coefficients of x means making equal the number of x on each side of the identity.

Equate coefficients of x $\quad -6 = -2p$

$3 = p$

Equate constants $\quad 2 = p^2 + q$

$2 = 9 + q$

$-7 = q$

$p = 3$ and $q = -7$

Completing the square

Example 2.21

Work out the values of a, b and c such that $2x^2 + bx + 5 = a(x - 3)^2 + c$.

Solution

Expand the bracket $\quad 2x^2 + bx + 5 = a(x^2 - 6x + 9) + c$

$$= ax^2 - 6ax + 9a + c$$

Equate coefficients of x^2 $\quad 2 = a$

Equate coefficients of x $\quad b = -6a$

$$b = -12$$

Equate constants $\quad 5 = 9a + c$

$$5 = 18 + c$$

$$-13 = c$$

$$a = 2,\ b = -12 \text{ and } c = -13$$

Example 2.22

Work out the values of a, b and c such that $3x^2 + 5x - 1 = a(x + b)^2 + c$.

Solution

Expand the bracket $\quad 3x^2 + 5x - 1 = a(x^2 + 2bx + b^2) + c$

$$= ax^2 + 2abx + ab^2 + c$$

Equate coefficients of x^2 $\quad 3 = a$

Equate coefficients of x $\quad 5 = 2ab$

$$5 = 6b$$

$$\frac{5}{6} = b$$

Equate constants $\quad -1 = ab^2 + c$

$$-1 = 3 \times \left(\frac{5}{6}\right)^2 + c$$

$$-1 = 3 \times \frac{25}{36} + c$$

$$-1 = \frac{25}{12} + c$$

$$-\frac{37}{12} = c$$

$$a = 3,\ b = \frac{5}{6} \text{ and } c = -\frac{37}{12}$$

Note

Comparing coefficients is a useful technique which can be applied to any polynomial.

An alternative method when rewriting a quadratic in the form $a(x + b)^2 + c$, is to use the technique given in Chapter 4.

Exercise 2F

① Work out the values of a and b such that $x^2 + 8x + 10 = (x + a)^2 + b$.

② Work out the values of c and d such that $x^2 - cx + 7 = (x - 1)^2 + d$.

③ Work out the values of p and q such that $x^2 - 12x - 4 = (x - p)^2 + q$.

④ Work out the values of a and b such that $x^2 + 5x - 2 = (x + a)^2 + b$.

⑤ Work out the values of p and q such that $5 + 4x - x^2 = p - (x - q)^2$.

⑥ Work out the values of c and d such that $2 - x - x^2 = c - (x + d)^2$.

⑦ Work out the values of a, b and c such that $2x^2 + bx + 5 = a(x + 2)^2 + c$.

⑧ Work out the values of a, b and c such that $5x^2 + 30x + 10 = a(x + b)^2 + c$.

⑨ Work out the values of p, q and r such that $3x^2 - 12x + 14 = p(x + q)^2 + r$.

⑩ Work out the values of a, b and c such that $3x^2 - bx + 1 = a(x - 4)^2 + c$.

⑪ Work out the values of a, b and c such that $6 + bx - 2x^2 = c - a(x - 1)^2$.

⑫ Work out the values of p, q and r such that $5 - 12x - 2x^2 = p - q(x + r)^2$.

⑬ (i) Work out the values of a and b such that $x^2 - 8x + 20 = (x - a)^2 + b$.

(ii) Hence make x the subject of $y = x^2 - 8x + 20$

⑭ (i) Work out the values of p, q and r such that $3x^2 + 6x + 1 = p(x + q)^2 + r$.

(ii) Hence make x the subject of $y = 3x^2 + 6x + 1$

FUTURE USES

The ability to manipulate algebraic expressions and solve equations is fundamental to much of Pure Mathematics. Factorisation and completing the square will be revisited in Chapter 4 where you will be solving quadratic equations and inequalities.

You will use the skills learnt in this chapter extensively throughout this book. If you choose to study Mathematics at a higher level you will find the techniques introduced here invaluable.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- factorise an algebraic expression using no more than two brackets
- rearrange a formula to make a different letter the subject
  - when the new subject occurs once
  - when the new subject appears more than once
- simplify algebraic fractions connected by any of the symbols +, −, ×, or ÷
- solve linear equations containing algebraic fractions
- write a quadratic expression in the form $a(x + b)^2 + c$.

KEY POINTS

1 When factorising a quadratic expression you need to write it as a product using brackets.

2 When changing the subject of an equation the new subject should be on its own on the left-hand side.

3 When simplifying an algebraic fraction involving addition or subtraction you need to find a common denominator.

4 When solving an equation involving fractions you start by multiplying through by the least common multiple of all the denominators to eliminate the fractions.

5 Quadratic expressions can be written in the form $a(x + b)^2 + c$.

3 Algebra III

Others have done it before me. I can, too.
Corporal John Faunce (American soldier)

1 Function notation

Here is a flow chart.

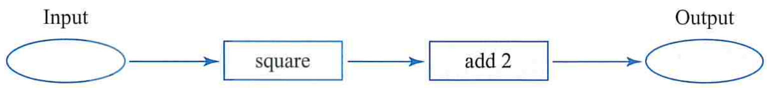


Figure 3.1

For an input of 5, $5 \rightarrow 25 \rightarrow 27$ the output is 27

For an input of −2, $-2 \rightarrow 4 \rightarrow 6$ the output is 6

For an input of x, $x \rightarrow x^2 \rightarrow x^2 + 2$ the output is $x^2 + 2$

This leads to the use of function notation $f(x) = x^2 + 2$

For an input of 5,

$$\begin{aligned} f(5) &= 5^2 + 2 \\ &= 25 + 2 \\ &= 27 \end{aligned}$$

For an input of −2,

$$\begin{aligned} f(-2) &= (-2)^2 + 2 \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

A **function** must have a unique output for every input.

Consequently $f(x) = \pm x^2$ is not a function.

Discussion point

➜ Which of the following are functions?

(i) $f(x) = (\pm x)^2$

(ii) $f(x) = (1 \pm x)^2$

Example 3.1

$f(x) = 10 - 4x$ and $g(x) = x^3$

(i) Evaluate $f(-1)$ and $g\left(\frac{1}{2}\right)$.

(ii) Write down an expression for $f(3x)$.

(iii) Solve $g(x) = -64$

Solution

(i) $f(-1) = 10 - 4(-1)$
$= 10 + 4$
$= 14$

$g\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3$
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

(ii) $f(3x) = 10 - 4(3x)$
$= 10 - 12x$

(iii) $g(x) = -64$
$x^3 = -64$
$x = \sqrt[3]{-64}$
$= -4$

Exercise 3A

① $f(x) = 2x - 1$ and $g(x) = x^2 + 2x$
Work out the value of
(i) $f(-4)$ (ii) $f(0.6)$ (iii) $g(3)$
(iv) $g(-1)$ (v) $f(0)$ (vi) $g(0)$.

② $f(x) = 3x^2$ and $g(x) = \frac{6}{x}$
Work out the value of
(i) $f(2)$ (ii) $f(-5)$ (iii) $g(2)$
(iv) $g(-1.5)$ (v) $g\left(\frac{1}{2}\right)$ (vi) $g\left(-\frac{2}{3}\right)$.

③ $f(x) = (2x - 1)^2$ and $g(x) = 2x + 1$. Work out the value of
(i) $f(0)$ (ii) $g(-2)$ (iii) $f(0.5)$
(iv) $f\left(-\frac{1}{4}\right)$ (v) $g\left(-\frac{1}{2}\right)$ (vi) $g(1.6)$.

④ $f(x) = 8 - 3x$ and $g(x) = 4(x + 3)$. Solve
(i) $f(x) = 0$ (ii) $g(x) = 20$ (iii) $f(x) = g(x)$.

⑤ $h(x) = 3x - 2$
Write down expressions, giving answers in the simplest form, for
(i) $h(2x)$ (ii) $h(x + 1)$ (iii) $h(x^2)$.

⑥ $f(x) = (x - 1)^2$
Write down expressions, giving answers in the simplest form, for
(i) $f(x^2)$ (ii) $[f(x)]^2$ (iii) $(f(x + 1))^2$.

⑦ $f(x) = x^2 + 5x - 1$

Write down expressions, giving answers in the simplest form, for

(i) $f(3x)$ (ii) $f(x - 2)$.

⑧ $g(x) = \frac{x + 6}{2x}$

(i) Work out the value of g(3). (ii) Solve $g(x) = 3$.

(iii) Solve $g(2x) = 1$.

2 Domain and range of a function

Discussion point

→ Why is it not possible for $x = 0$ to be in the domain of $g(x) = \frac{1}{x}$?

! Take care to check whether the domain uses the symbols $>$ and $<$ or $\geqslant$ and $\leqslant$. If the domain includes the option of equality then so must the range. If the domain excludes a certain point or points, then look for exclusions in the range.

Domain of f(x)

The set of input values is the domain of $f(x)$.

When a function is defined it will include a domain. If a domain is not stated it can be assumed that the domain is all real values of x.

$f(x) = 2x - 3 \quad x > 1$ The domain is $x > 1$

$g(x) = \frac{1}{x} \quad x \neq 0$ The domain is all real values of x apart from 0

$h(x) = x^2$ The domain is all real values of x.

Range of f(x)

The set of output values is the range of $f(x)$.

The range is usually dependent upon the domain (an exception is $f(x) = 2$).

The range is given as a set of $f(x)$ values as shown in the following examples.

Example 3.2

$f(x) = x^2$ for all real values of x.

Write down the range of $f(x)$.

Solution

The value of x^2 is positive or zero for all real values of x.

Range is $f(x) \geqslant 0$.

Note

Domain and range can be seen on a sketch graph of $f(x) = x^2$.

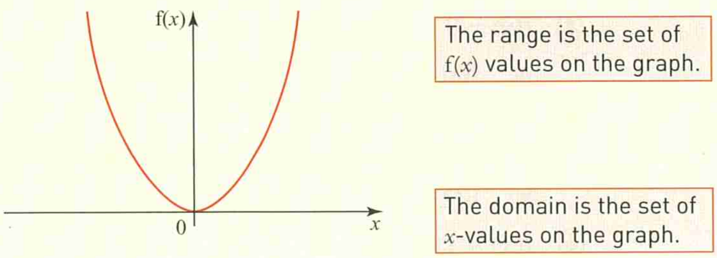


Figure 3.2

ACTIVITY 3.1

Sketch the graph of $g(x) = \frac{1}{x}$ for $x \neq 0$. Use the graph to identify the range of $g(x)$.

Example 3.3

$f(x) = 6 - 4x \qquad -2 \leqslant x \leqslant 3$

Write down the range of $f(x)$.

Solution

A sketch of $f(x) = 6 - 4x$ for the given domain is

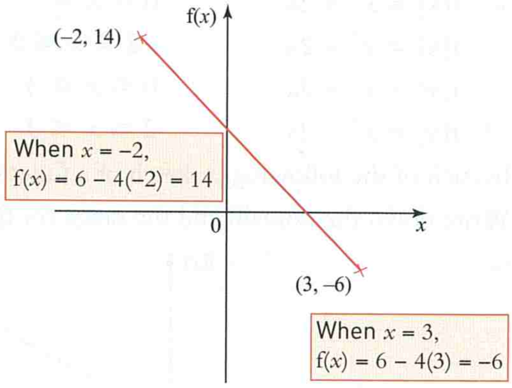


Figure 3.3

Range is $-6 \leqslant f(x) \leqslant 14$.

Exercise 3B

① Write down the range of $f(x)$ in each of the following.

(i) $f(x) = 3x \qquad x < 2$

(ii) $f(x) = x + 4 \qquad x \geqslant 1$

(iii) $f(x) = 2x + 4 \qquad x \geqslant -1$

(iv) $f(x) = 10 - x \qquad x \leqslant 4$

② Write down the range of $f(x)$ in each of the following.

(i) $f(x) = 2x \qquad 1 \leqslant x \leqslant 5$

(ii) $f(x) = x - 3 \qquad 0 < x < 10$

(iii) $f(x) = 5 - 2x \qquad x \geqslant -3$

(iv) $f(x) = 3 - 4x \qquad -2 \leqslant x < 3$

③ Write down the range of $f(x)$ in each of the following.

(i) $f(x) = \dfrac{x + 5}{2} \qquad 0 \leqslant x \leqslant 5$

(ii) $f(x) = \dfrac{2x - 3}{4} \qquad -2 \leqslant x \leqslant 2$

(iii) $f(x) = \dfrac{3 - 2x}{3} \qquad -3 \leqslant x \leqslant 5$

(iv) $f(x) = \dfrac{1 - 3x}{2} \qquad -3 \leqslant x \leqslant 5$

④ Write down the range of $f(x)$ in each of the following.

(i) $f(x) = x^2 \qquad -2 \leqslant x < 2$

(ii) $f(x) = x^2 \qquad 0 < x < 4$

(iii) $f(x) = x^3 \qquad x \geqslant 0$

(iv) $f(x) = x^3 \qquad -1 \leqslant x \leqslant 3$

⑤ Write down the range of f(x) in each of the following.

(i) $f(x) = 2x^2 - 3 \qquad 0 \leqslant x \leqslant 4$

(ii) $f(x) = 3x^2 - 2 \qquad 0 \leqslant x \leqslant 4$

(iii) $f(x) = 3 - 2x^2 \qquad -1 \leqslant x \leqslant 2$

(iv) $f(x) = 2 - 3x^2 \qquad -1 \leqslant x \leqslant 2$

⑥ Write down the range of f(x) in each of the following.

(i) $f(x) = x^2 + 2x \qquad 0 \leqslant x \leqslant 3$

(ii) $f(x) = x^2 + 2x \qquad -2 \leqslant x \leqslant 3$

(iii) $f(x) = x^2 - 2x \qquad 0 \leqslant x \leqslant 3$

(iv) $f(x) = x^2 - 2x \qquad 2 \leqslant x \leqslant 3$

⑦ In each of the following, a sketch of a function, f(x), is shown. Write down the domain and the range for f(x).

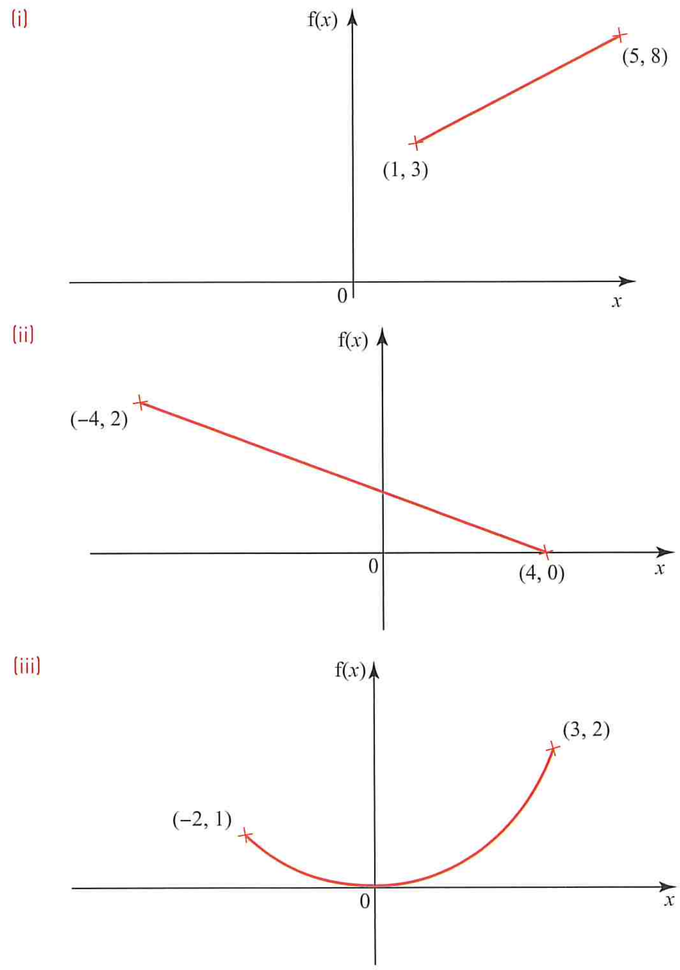


Figure 3.4

⑧ Sketch the graph of each of the following functions and write down the corresponding range.

(i) $f(x) = 3x - 2$ for $-1 \leqslant x \leqslant 3$

(ii) $f(x) = 2 - 3x$ for $-1 \leqslant x \leqslant 2$

(iii) $f(x) = x^2 + 3$ for $-2 \leqslant x \leqslant 2$

(iv) $f(x) = 4 - x^2$ for $-2 \leqslant x \leqslant 3$

Unlike when adding or multiplying two functions, the order in which you apply the functions is critical.

3 Composite functions

A **composite function** occurs when two or more functions act in succession.

$fg(x)$ means that the function next to x, which is g in this case, is the one that is applied first. Using $f(x) = x^2$ and $g(x) = x + 2$, it is often easier to think of this in words initially, rather than symbols.

$fg(x) = f[g(x)]$ which means that you first apply the function g to x.
$g(x) = (x + 2)$ so you now have $f(x + 2)$.

In this example the function f tells you to square what you have in the brackets, giving $(x + 2)^2$ so $fg(x) = (x + 2)^2$.

Similarly $gf(x) = x^2 + 2$

Discussion point

→ Using the functions above, are there any values of x for which $fg(x) = gf(x)$?

Combining functions using addition, multiplication or division

Using the functions above, if $f(x) = x^2$ and $g(x) = x + 2$, then

$f(x) + g(x) = x^2 + (x + 2) = x^2 + x + 2$

$f(x)g(x) = f(x) \times g(x) = x^2(x + 2) = x^3 + 2x^2$

$\dfrac{f(x)}{g(x)} = f(x) \div g(x) = \dfrac{x^2}{x + 2}$.

Example 3.4

Express $(5x + 6)^4$ in the form $fg(x)$, stating the expressions corresponding to $f(x)$ and $g(x)$.

Solution

Starting with x gives the flow chart $x \rightarrow 5x + 6 \rightarrow (5x + 6)^4$

This is the same as $x \rightarrow g(x) \rightarrow fg(x)$ ← g must be applied first.

giving $g(x) = 5x + 6$, $f(x) = x^4$

so $fg(x) = (5x + 6)^4$.

Example 3.5

Given that $f(x) = 2x - 3$, $g(x) = x^2$ and $h(x) = \frac{1}{x}$, write the following as functions of x.

(i) $fg(x)$ (ii) $gf(x)$

(iii) $fgh(x)$ (iv) $f^2(x)$

Solution

(i) $fg(x) = f[g(x)]$
$= f(x^2)$
$= 2x^2 - 3$

(ii) $gf(x) = g[f(x)]$
$= g(2x - 3)$
$= (2x - 3)^2$

(iii) $fgh(x) = fg[h(x)]$
$= f\left[g\left(\frac{1}{x}\right)\right]$
$= f\left(\frac{1}{x^2}\right)$
$= \frac{2}{x^2} - 3$

(iv) $f^2(x) = f[f(x)]$
$= f(2x - 3)$
$= 2(2x - 3) - 3$
$= 4x - 6 - 3$
$= 4x - 9$

ACTIVITY 3.2

For the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$, use your calculator to evaluate f(4), g(4), fg(4) and gf(4) and say what you notice.
What happens when you replace 4 by −4?

Exercise 3C

① Express $(3 + x)^2$ in the form $fg(x)$, stating the expressions corresponding to $f(x)$ and $g(x)$.

② (i) Given that $f(x) = 2x - 1$ and $g(x) = x^3$, write

(a) $fg(x)$ and

(b) $gf(x)$.

(ii) Work out the values of

(a) fg(2) (b) gf(2)

(c) fg(−3) (d) gf(−3).

③ (i) Given that $f(x) = x^2$, $g(x) = \frac{1}{x}$ and $h(x) = 1 - x$, write the following as functions of x.

(a) $fg(x)$ (b) $fh(x)$

(c) $gf(x)$ (d) $hf(x)$

(ii) What do you notice?

④ The function $f(x) = (1 - x)^3$ is a composite function. Define $g(x)$ and $h(x)$ such that $f(x) = gh(x)$.

5. The function $f(x)$ is a composition of two functions $gh(x)$. Define $g(x)$ and $h(x)$, where $f(x)$ is

   (i) $\dfrac{3}{x-2}$ (ii) $\dfrac{x-2}{3}$

   (iii) $(3x-1)^2$ (iv) 2^{3x-1}.

6. The function $f(x)$ is a composition of two functions $uv(x)$. Define $u(x)$ and $v(x)$, where $f(x)$ is

   (i) $\sin 2x$ (ii) $\cos\dfrac{x}{2}$

   (iii) $\tan(x-30°)$ (iv) $\sin^2 x$.

7. The function $f(x)$ is a composition of three functions $pqr(x)$.

   Define $p(x)$, $q(x)$ and $r(x)$, where $f(x)$ is

   (i) $3(x-2)^4$

   (ii) $\dfrac{2x+3}{4}$.

Although there is theoretically no limit to the number of functions that you will be asked to combine, in an algebraic question it will usually be only two or three.

However, in a practical situation you are often required to perform a number of tasks in the correct order, as in the question below.

8. Daniel and Amanda are on holiday when their car gets a puncture and they need to change the wheel. Tasks here are not listed in the correct order.

   (a) Remove the wheel with the punctured tyre

   (b) Jack up the car

   (c) Open the boot

   (d) Put the jack and the wheel with the punctured tyre in the boot

   (e) Get the jack and the new wheel from the boot

   (f) Put the replacement wheel on the car

   (g) Close the boot

   There is a fixed order for performing these tasks. Treating each task as a function, write down the composite function for replacing the wheel. (Remember that the function performed first will appear at the end of the list).

4 Graphs of functions

Drawing or plotting a graph

If asked to draw a graph you should use graph paper. The axes should be numbered. The graph should be drawn passing through the points which have either been given or calculated.

On the next page is a drawing of the graph of $y = 2x + 1$ for values of x from -2 to 4. In this case the coordinates of the points have been calculated.

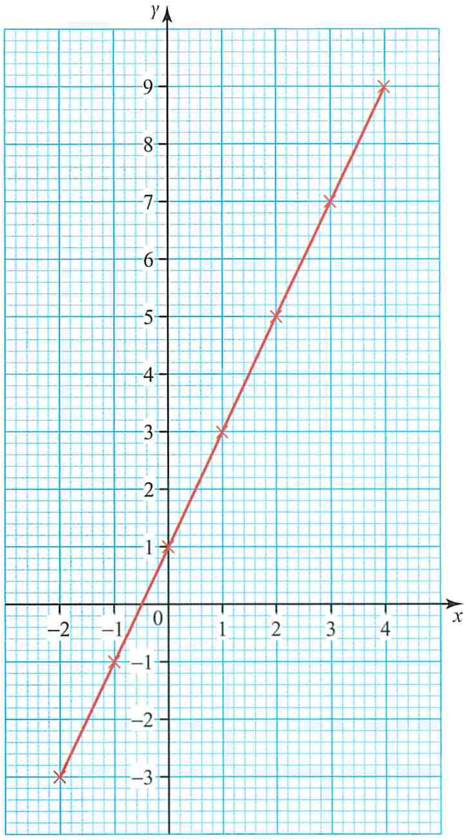


Figure 3.5

Note

In some cases you may meet the notation $f(x) = 2x + 1$ and then be told to draw the graph of $y = f(x)$. This is exactly the same instruction as 'Draw $y = 2x + 1$'.

Sketching a graph

If asked to sketch a graph you should *not* use graph paper. Axes should be drawn and only certain numbers need to be marked on the axes (e.g. points where the graph crosses the axes).

The correct shape of the graph should be shown and it should be in the correct position relative to the axes.

This means that the main features of the graph are shown although there is no requirement to plot points accurately.

Here is a sketch of the graph of $y = 2x + 1$

No domain (x value) was given so axes should be drawn to show all four quadrants.

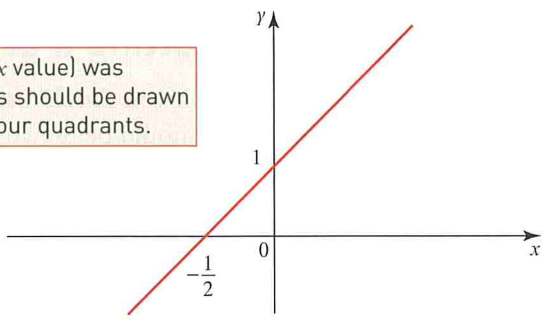


Figure 3.6

5 Graphs of linear functions

The gradient of a line

Discussion point

➜ What information do you need to have in order to fix the position of a line?

In mathematics the word *linear* refers to a straight line. The slope of a line is measured by its *gradient* and the letter m is often used to represent this.

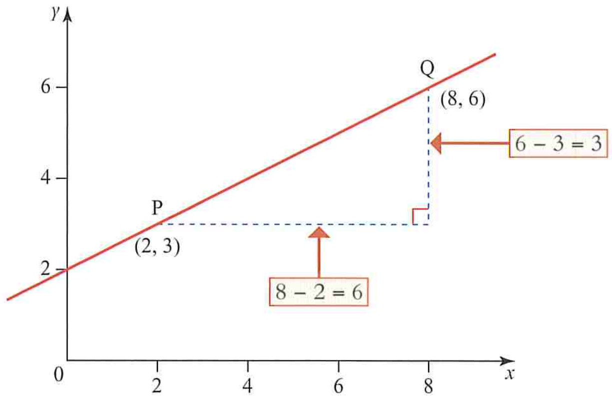


Figure 3.7

$$\text{gradient} = \frac{\text{change in } y\text{-coordinate from P to Q}}{\text{change in } x\text{-coordinate from P to Q}}$$

In Figure 3.7, gradient $= \frac{6-3}{8-2} = \frac{3}{6} = \frac{1}{2}$.

ACTIVITY 3.3

On each line in Figure 3.8, take any two points and use them to calculate the gradient of the line.

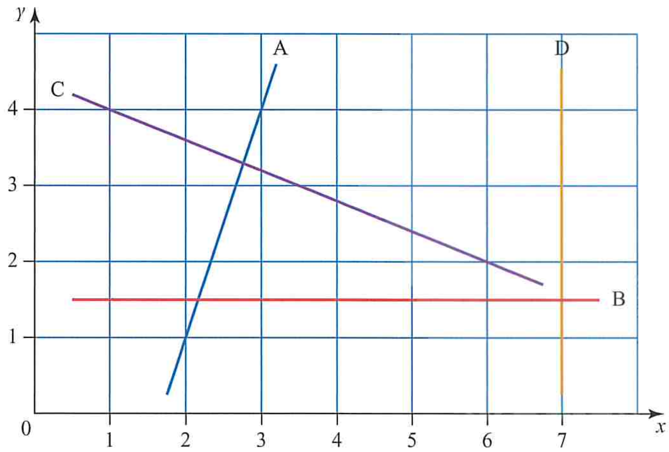


Figure 3.8

Discussion point

➜ Does it matter which point you call (x_1, y_1) and which (x_2, y_2)?

You can generalise the previous activity to find the gradient m of the line joining (x_1, y_1) to (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

You can easily tell by looking at a line if its gradient is positive, negative, zero or infinite.

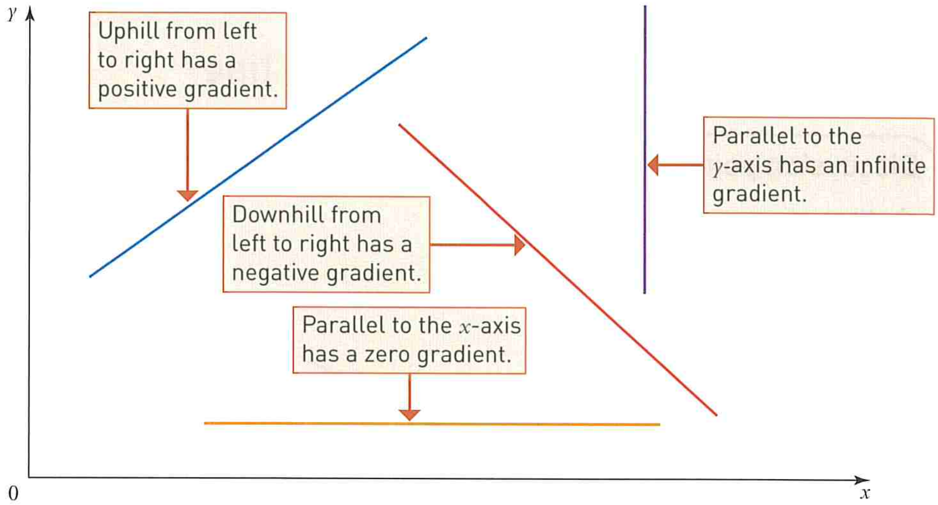


Figure 3.9

The equation of a straight line

Example 3.6

Work out the equation of the straight line with gradient 2 through the point with coordinates (0, 1).

Solution

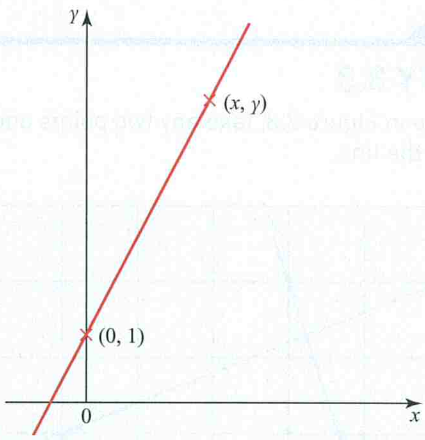


Figure 3.10

Take a general point (x, y) on the line, as shown in Figure 3.10, together with the point (0, 1) that you are given. The gradient of the line joining (0, 1) to (x, y) is given by

$$\text{gradient} = \frac{y-1}{x-0} = \frac{y-1}{x}.$$

Since you are given that the gradient of the line is 2,

$$\frac{y-1}{x} = 2 \quad \Rightarrow \quad y = 2x + 1$$

Since (x, y) is a general point on the line, this holds for any point on the line and is therefore the equation of the line.

This example can be generalised to give the result that the equation of the line with gradient m cutting the y-axis at the point $(0, c)$ is

$$\frac{y - c}{x - 0} = m$$

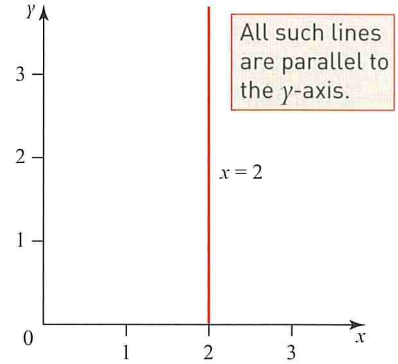
$$\Rightarrow \quad y = mx + c.$$

This is a well-known standard form for the equation of a straight line.

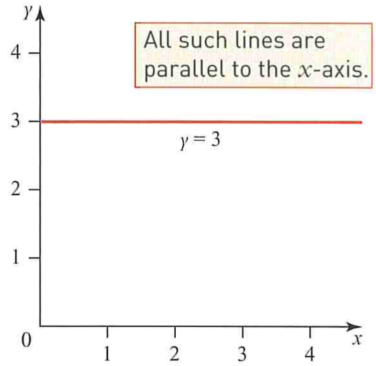
Drawing or sketching a line given its equation

There are several standard forms for the equation of a straight line. When you need to draw or sketch a line, look at its equation and see if it fits one of these.

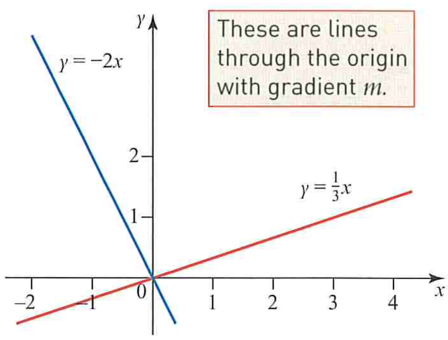
Equations of the form $x = a$



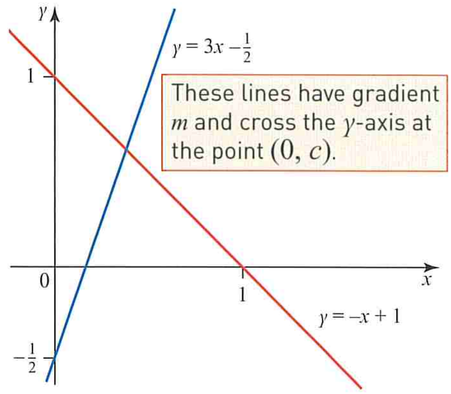
Equations of the form $y = b$



Equations of the form $y = mx$



Equations of the form $y = mx + c$



Equations of the form $px + qy + r = 0$

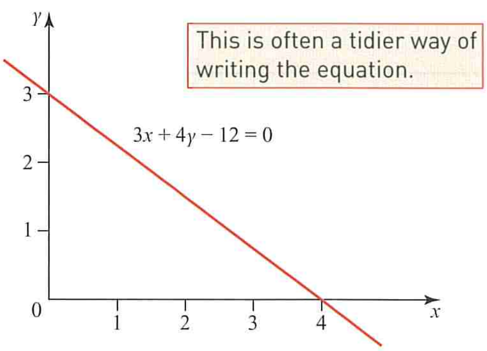


Figure 3.11

Graphs of linear functions

Graphs of equations in the form $px + qy + r = 0$ will usually be sketched by finding the coordinates of the points where the line crosses the x- and y-axes.

Discussion point

(i) Rearrange the equation $3x + 4y - 12 = 0$ into the form $\frac{x}{a} + \frac{y}{b} = 1$.

(ii) What are the values of a and b?

(iii) What do these numbers represent?

Example 3.7

Sketch the lines $y = -2$, $y = 3x - 2$ and $x + 3y - 9 = 0$ on the same axes.

Solution

The line $y = -2$ is parallel to the x-axis and passes through $(0, -2)$.

The line $y = 3x - 2$ has gradient 3 and passes through $(0, -2)$.

To sketch the line $x + 3y - 9 = 0$ find two points on it.

$x = 0$	$\Rightarrow$	$3y - 9 = 0$	$\Rightarrow$	$y = 3$	$(0, 3)$ is on the line
$y = 0$	$\Rightarrow$	$x - 9 = 0$	$\Rightarrow$	$x = 9$	$(9, 0)$ is on the line

Figure 3.12 shows the three lines.

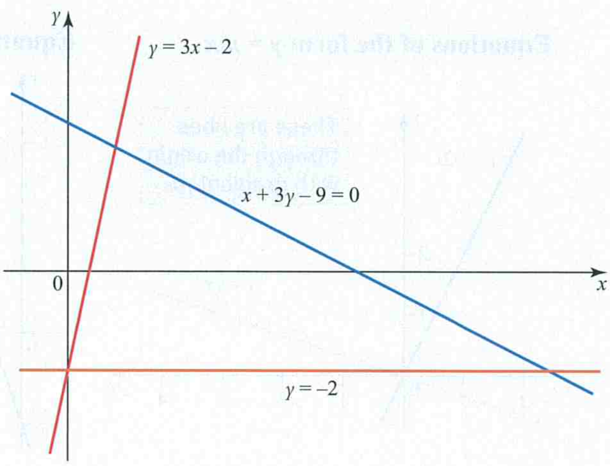


Figure 3.12

Exercise 3D

① For each of the following pairs of points, A and B, calculate the gradient of the line AB.

(i)	A(4, 3)	B(8, 11)	(ii)	A(3, 4)	B(0, 13)
(iii)	A(5, 3)	B(10, –8)	(iv)	A(–6, –14)	B(1, 7)
(v)	A(6, 0)	B(8, 15)	(vi)	A(–2, –4)	B(3, 9)
(vii)	A(–3, –6)	B(2, –7)	(viii)	A(4, 7)	B(7, –4)

In the following questions mark the coordinates of all points of intersection with the axes.

② Sketch these lines.

(i) $x = 5$ (ii) $y = -3$ (iii) $x = 0$ (iv) $y = 0$

③ Sketch these lines.

(i) $y = 4x$ (ii) $y = -3x$ (iii) $y = 4 + x$ (iv) $y = -3 + x$

④ Sketch these lines.

(i) $y = 2x + 3$ (ii) $y = 2x - 3$ (iii) $y = 2 + 3x$ (iv) $y = 2 - 3x$

⑤ Sketch these lines.

(i) $y = \frac{1}{2}x - 1$ (ii) $y = \frac{1}{3}x + \frac{2}{3}$

(iii) $y = 2 - \frac{1}{2}x$ (iv) $y = 3 - \frac{2x}{3}$

⑥ Sketch these lines.

(i) $x + 2y = 5$ (ii) $3x - y = 4$

(iii) $2x + y = 0$ (iv) $x - 2y = 0$

⑦ Sketch these lines

(i) $x + y - 1 = 0$ (ii) $2x + y - 4 = 0$

(iii) $x - 3y + 6 = 0$ (iv) $y - 3x + 9 = 0$

⑧ Sketch these lines.

(i) $\frac{x}{2} - \frac{y}{3} - 1 = 0$ (ii) $\frac{x}{3} - \frac{y}{2} - 1 = 0$

(iii) $\frac{3x}{2} - \frac{2y}{3} - 1 = 0$ (iv) $\frac{2x}{3} - \frac{3y}{2} - 1 = 0$

⑨ A printer quotes the cost £C of printing n business cards as $C = 60 + 0.06n$.

(i) Work out the cost of printing

(a) 500 cards

(b) 5000 cards

and the cost per card in each case.

(ii) The cost is made up of a fixed cost for setting up the printer and a cost per card printed. State the cost of each of these.

(iii) Sketch the graph of C against n.

If you have access to a graphic calculator, you can use it to check your results. Alternatively, check your answers with a free online graphing resource.

Prior knowledge

Students should be confident when manipulating algebraic expressions, including algebraic fractions, as covered in Chapter 2.

6 Finding the equation of a line

The simplest way of finding the equation of a straight line depends on what information you have been given.

Given the gradient, m, and the point of intersection $(0, c)$ with the y-axis

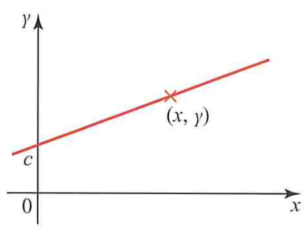


Figure 3.13

The gradient of the line joining $(0, c)$ to (x, y) is given by

$$m = \frac{y - c}{x - 0}$$

$$\Rightarrow y = mx + c$$

Given the gradient, m, and the coordinates (x_1, y_1) of a general point on the line

Take the general point (x, y) on the line, as shown in Figure 3.14.

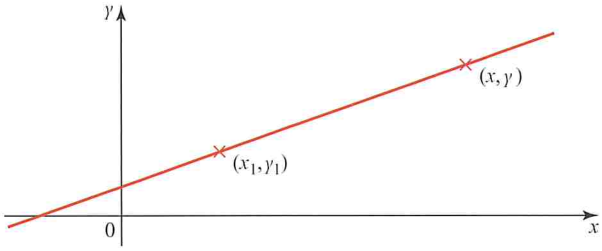


Figure 3.14

The gradient m of the line joining (x_1, y_1) to (x, y) is given by

$$m = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow y - y_1 = m(x - x_1).$$

This is a standard result, and one you will find very useful.

Example 3.8

Work out the equation of the line with gradient 2 which passes through the point (–1, 3).

Solution

Using $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 3 = 2(x - (-1))$$

$$\Rightarrow y - 3 = 2x + 2$$

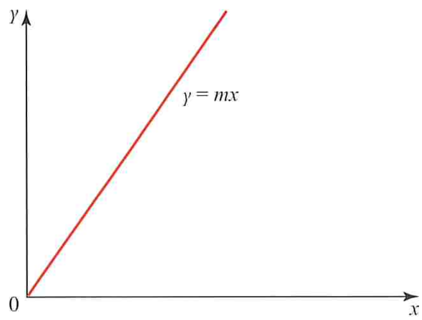
$$\Rightarrow y = 2x + 5$$

In the formula

$$y - y_1 = m(x - x_1)$$

two positions of the point (x_1, y_1) lead to results you have met already.

(x_1, y_1) is at $(0, 0)$ — (x_1, y_1) is at $(0, c)$



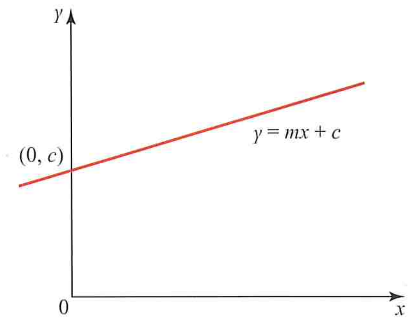


Figure 3.15

Given two points (x_1, y_1) and (x_2, y_2)

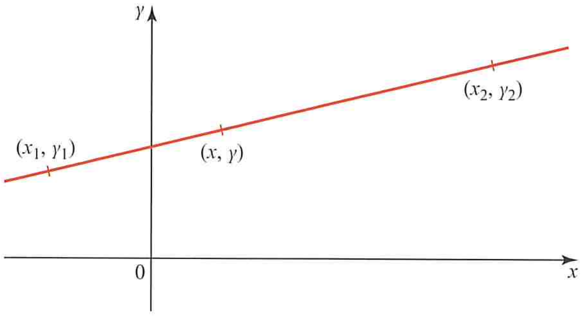


Figure 3.16

The two points are used to work out the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This value is then substituted in the equation.

$y - y_1 = m(x - x_1)$.

This gives

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Rearranging this gives

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \text{or}$$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 3.9

Work out the equation of the line joining (–1, 4) to (2, –3).

Solution

Let (x_1, y_1) be (–1, 4) and (x_2, y_2) be (2, –3).

Substituting these values in $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

gives $\dfrac{y - 4}{(-3) - 4} = \dfrac{x - (-1)}{2 - (-1)}$

$$\Rightarrow \quad \frac{y - 4}{(-7)} = \frac{x + 1}{3}$$

$$\Rightarrow \quad 3(y - 4) = (-7)(x + 1)$$

$$\Rightarrow \quad 7x + 3y - 5 = 0$$

Applying the different techniques

Example 3.10

Work out the equations of the lines **(a)** to **(c)** in Figure 3.17.

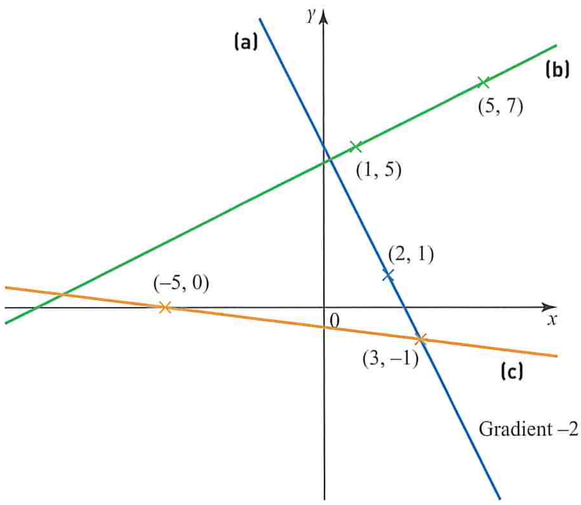


Figure 3.17

Solution

Line **(a)** has a gradient of −2 and passes through the point (2, 1).

Using $y - y_1 = m(x - x_1)$

$y - 1 = -2(x - 2)$

$\Rightarrow y - 1 = -2x + 4$

$\Rightarrow y = -2x + 5$

This may also be written as $2x + y = 5$ or $2x + y - 5 = 0$

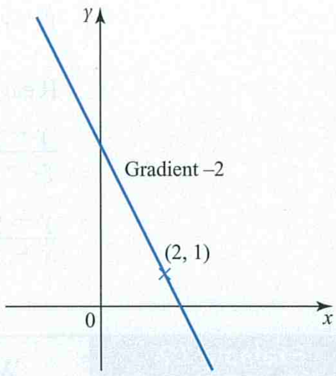


Figure 3.18

Line **(b)** passes through the points (1, 5) and (5, 7).

Using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the gradient

$m = \dfrac{7 - 5}{5 - 1} = 0.5$

Using $y - y_1 = m(x - x_1)$ and the point (1, 5)

$y - 5 = 0.5(x - 1)$

$\Rightarrow y - 5 = 0.5x - 0.5$

$\Rightarrow y = 0.5x + 4.5$

Avoiding a decimal in the answer this could also be given as $2y = x + 9$ or $x - 2y + 9 = 0$.

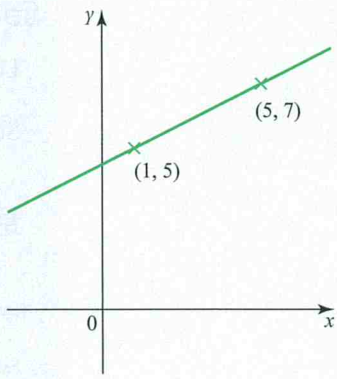


Figure 3.19

Line **(c)** passes through the points (−5, 0) and (3, −1).

Let (x_1, y_1) be (−5, 0) and (x_2, y_2) be (3, −1).

Substituting these values in

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

gives $\frac{y - 0}{-1 - 0} = \frac{x - (-5)}{3 - (-5)}$

$$\Rightarrow \frac{y}{-1} = \frac{x + 5}{8}$$

$$\Rightarrow 8y = -x - 5$$

$$\Rightarrow x + 8y + 5 = 0$$

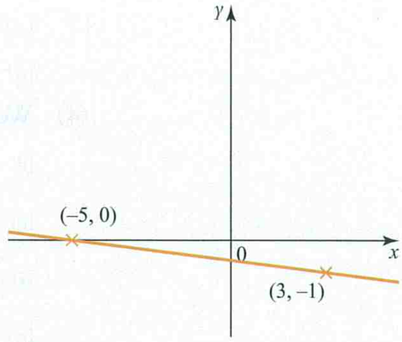


Figure 3.20

Exercise 3E

① Work out the equations of the lines **(i)** – **(v)** in this diagram.

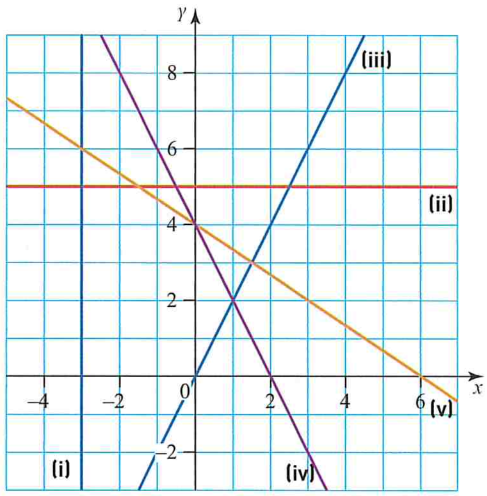


Figure 3.21

② Work out the equations of the lines **(i)** – **(v)** in this diagram.

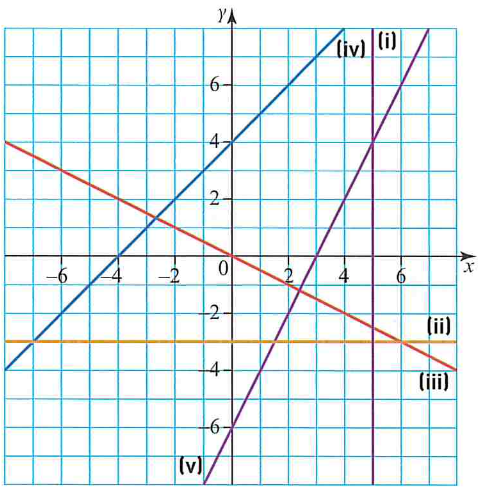


Figure 3.22

③ Work out the equations of these lines.

(i) Gradient 3 and passing through $(2, -1)$

(ii) Gradient 2 and passing through $(0, 0)$

(iii) Gradient 3 and passing through $(2, -7)$

(iv) Gradient 4 and passing through $(4, 0)$

④ Work out the equations of these lines.

(i) Gradient $\frac{1}{3}$ and passing through $(3, 1)$

(ii) Gradient $\frac{2}{5}$ and passing through $(-4, -10)$

(iii) Gradient $-\frac{3}{2}$ and passing through $(1, -2)$

(iv) Gradient $-\frac{1}{2}$ and passing through $(0, 6)$

⑤ Work out the equation of the line AB in each of these cases.

(i) A(2, 0) B(3, 1)

(ii) A(3, −1) B(0, 4)

(iii) A(2, −3) B(3, −2)

⑥ Work out the equation of the line AB in each of these cases.

(i) A(−1, 3) B(4, 0)

(ii) A(3, −5) B(10, −6)

(iii) A(−1, −2) B(−4, −8)

⑦ A taxi journey costs £2 plus 80 pence per mile. Use £C to represent the total cost of the journey and m miles to represent the total distance travelled.

(i) Write down an equation giving C in terms of m.

(ii) How much would a journey of 4 miles cost?

(iii) How far could I travel if I only had £10?

⑧ A junior school is ordering exercise books for their students and is working on the assumption that most students will only use 8 books during the year, but they want to order an additional 100. Let N represent the number of books to be ordered and let s be the number of students enrolled for the year.

(i) Write down an equation giving N in terms of s.

(ii) The exercise books cost £1.50 each. If there are 240 students that year, what would be the total cost of the books?

(iii) The school budget for exercise books is only £3000. How could the order be amended?

7 Graphs of quadratic functions

Discussion point

➜ What is the difference between a quadratic function and a quadratic equation?

ACTIVITY 3.4

Copy and complete the table of values and draw the graph of $y = x^2 - 5$ for values of x from −3 to 4.

x	−3	−2	−1	0	1	2	3	4
y	4			−5		−1	4	

ACTIVITY 3.5

Copy and complete the table of values and draw the graph of $y = 4x - x^2$ for values of x from -2 to 6.

x	-2	-1	0	1	2	3	4	5	6
y	-12	-5			4				-12

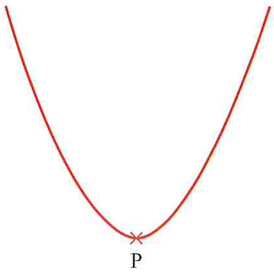


Figure 3.23

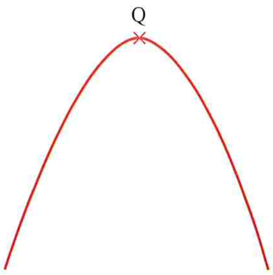


Figure 3.24

The shape of the graph of $y = ax^2 + bx + c$ is a parabola.

The sign of the coefficient of x^2 determines the direction of the curve.

$a > 0$

P is the lowest point on the graph in Figure 3.23.

P is the vertex.

$a < 0$

Q is the highest point on the graph in Figure 3.24.

Q is the vertex.

Symmetry

Quadratic graphs have a line of symmetry when drawn using an appropriate domain.

Here is the graph of $y = x^2 - 2x - 3$ for domain $-2 \leqslant x \leqslant 4$

The vertex is $(1, -4)$.

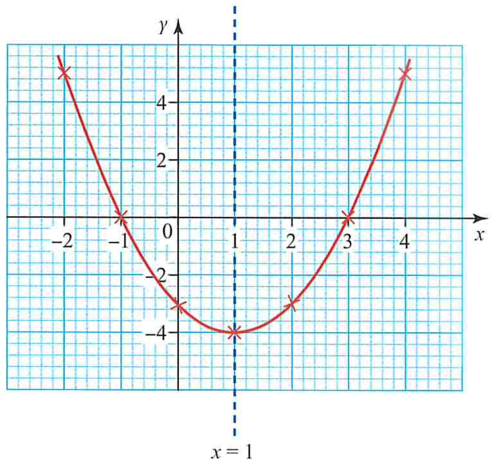


Figure 3.25

The line of symmetry has equation $x = 1$ and passes through the vertex.

Here is a sketch of the graph of $y = 9 - x^2$ for domain $-4 \leqslant x \leqslant 4$

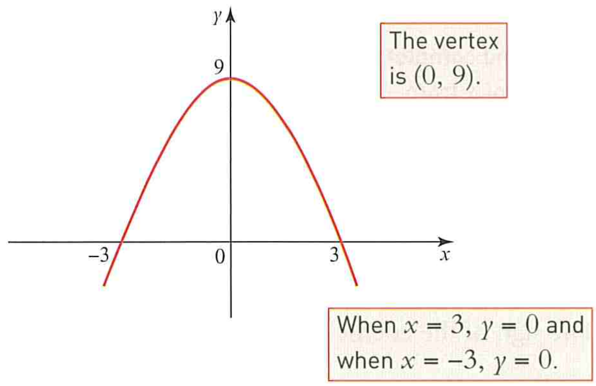


Figure 3.26

The line of symmetry is the y-axis and passes through the vertex.

Prior knowledge

Students are expected to be familiar with the technique of completing the square, used to find the vertex and the line of symmetry of a quadratic graph, from GCSE. This is used in the following example.

This technique is covered in more detail in the next chapter.

Example 3.11

For the graph of $y = x^2 + 6x + 11$, state

(i) the vertex

(ii) the equation of the line of symmetry

(iii) the coordinates of the point where the graph intersects the y-axis.

Sketch the graph.

Solution

Write the quadratic expression in the form $(x + a)^2 + b$.

$$x^2 + 6x + 11 \equiv (x + a)^2 + b$$
$$\equiv x^2 + 2ax + a^2 + b$$

Equate coefficients of x $\quad 6 = 2a$

$$3 = a$$

Equate constants $\quad 11 = a^2 + b$

$$11 = 9 + b$$
$$2 = b$$
$$y = x^2 + 6x + 11$$
$$= (x + 3)^2 + 2$$

Equate coefficients of x means making equal the number of x on each side of the identity.

$(x + 3)^2$ is always positive or zero.

The least value of $(x + 3)^2 + 2$ is 2 and this occurs when $x = -3$.

(i) The vertex is $(-3, 2)$.

(ii) The equation of the line of symmetry is $x = -3$.

(iii) When $x = 0$, $y = 11$, coordinates of y-axis intercept are $(0, 11)$.

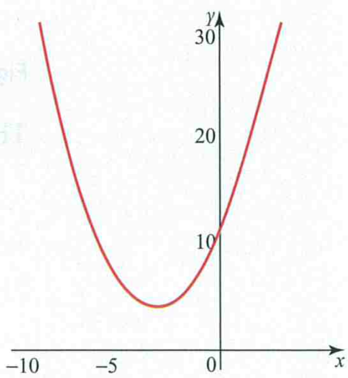


Figure 3.27

Exercise 3F

① Choose an equation from the list below to fit each of the quadratic curves **(i)** and **(ii)**.

$y = x^2 - 2x + 4$

$y = 5 - x^2$

$y = 3x - x^2$

$y = x^2 - 2x - 3$

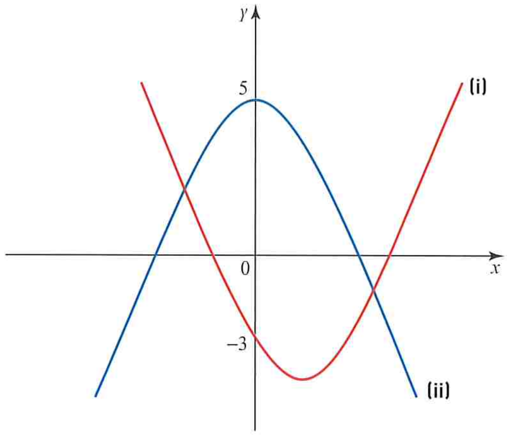


Figure 3.28

② Choose an equation from the list below to fit each of the quadratic curves **(i)** and **(ii)**.

$y = x^2 + 3x + 4$

$y = 4 - 7x - 2x^2$

$y = x^2 + 2x$

$y = 4x - x^2$

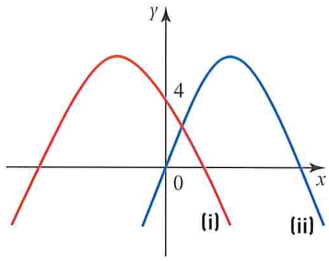


Figure 3.29

③ (i) For the graph of $y = x^2 + 2x + 3$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Sketch the graph.

④ (i) For the graph of $y = x^2 - 4x + 5$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Sketch the graph.

⑤ (i) For the graph of $y = x^2 - 6x + 7$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Sketch the graph.

⑥ (i) For the graph of $y = x^2 - 3x - 4$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Use trial and improvement to work out the coordinates of the points where the curve intersects the x-axis.

(iii) Sketch the graph.

⑦ (i) Show that $x^2 - 6x + 10 = (x - 3)^2 + 1$.

(ii) Write down the vertex of the graph of $y = x^2 - 6x + 10$.

(iii) Write down the equation of the line of symmetry of $y = x^2 - 6x + 10$.

(iv) Write down the coordinates of the points where the graph of $y = x^2 - 6x + 10$ intersects the coordinate axes.

(v) Sketch the graph of $y = x^2 - 6x + 10$.

⑧ (i) Show that $6 - 4x - x^2 = 10 - (x + 2)^2$.

(ii) Write down the vertex of the graph of $y = 6 - 4x - x^2$.

(iii) Write down the equation of the line of symmetry of $y = 6 - 4x - x^2$.

(iv) Write down the coordinates of the points where the graph of $y = 6 - 4x - x^2$ intersects the y-axis.

(v) Sketch the graph of $y = 6 - 4x - x^2$.

⑨ A quadratic curve has its vertex at $(3, -7)$ and passes through $(0, 2)$. Work out the equation of the curve.

⑩ A car hire company is negotiating the price of n identical new cars with the manufacturer and offers a total price £P, where $P = 20000n - 200n^2$.

(i) What is the likely cost of one car?

(ii) Work out the cost of an order for

(a) 5 cars (b) 20 cars (c) 50 cars.

(iii) Work out the average cost per car when the number of cars ordered is

(a) 5 cars (b) 20 cars (c) 50 cars.

(iv) Suggest why the manufacturer may not want to sell 50 cars at once.

8 Inverse functions

Finding the inverse function

Consider the function f where $f(x) = x + 2$ for $x \in \{1, 2, 3\}$.

Under this function the domain $\{1, 2, 3\}$ maps onto the image set $\{3, 4, 5\}$.

It is possible to reverse this mapping and find the inverse function where each element of this image set $\{3, 4, 5\}$ is mapped back onto the corresponding member of the set $\{1, 2, 3\}$, i.e. the inverse of the function 'undoes' the effect of the function.

In the example above, where the function is given by $f(x) = x + 2$, the inverse function is given by $f^{-1}(x) = x - 2$

Example 3.12

Work out $f^{-1}(x)$ when $f(x) = 2x + 3$

Solution

When $f(x) = 2x + 3$ the steps used to build up the function are

$$x \xrightarrow{\times 2} 2x \xrightarrow{+3} 2x + 3 = f(x)$$

Reversing this gives

$$x \xleftarrow{\div 2} 2x \xleftarrow{-3} 2x + 3 = f(x)$$

i.e. the reverse steps are subtract 3 then divide by 2.

This shows that when $f(x) = 2x + 3$, then $f^{-1}(x) = \frac{x - 3}{2}$

An alternative method for finding the inverse

Writing the function in the previous example as $y = 2x + 3$.

Step 1: Interchange x and y to give $x = 2y + 3$

Step 2: Make y the subject of $x = 2y + 3$

Subtract 3 $\quad x - 3 = 2y$

Divide by 2 $\quad y = \frac{x - 3}{2}$

EXTENSION

Note

The graph of a function and its inverse is extension material and will not be assessed.

The graph of a function and its inverse

Below is a sketch of the graphs of $f(x) = 2x + 3$ and its inverse, using the same scale on both axes.

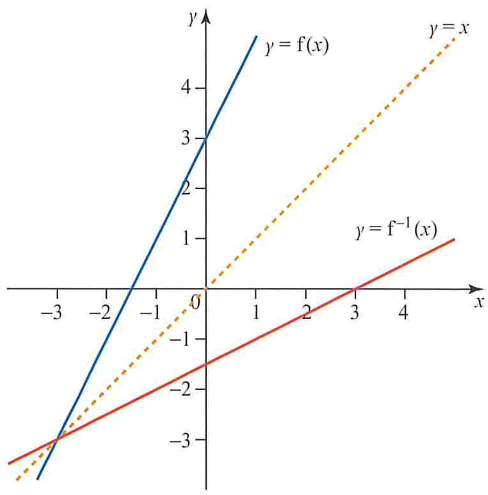


Figure 3.30

You can see from the graph that the function and its inverse are reflections of each other in the line $y = x$.

Inverse functions

A one-to-one function is one which has exactly one value of x for each value of y and one value of y for each value of x.

A function will only have an inverse if it is a one-to-one function in the domain given (the domain is the set of values of x).

For example, $y = x^2$ for $x \geqslant 0$ has an inverse $y = \sqrt{x}$ as shown below.

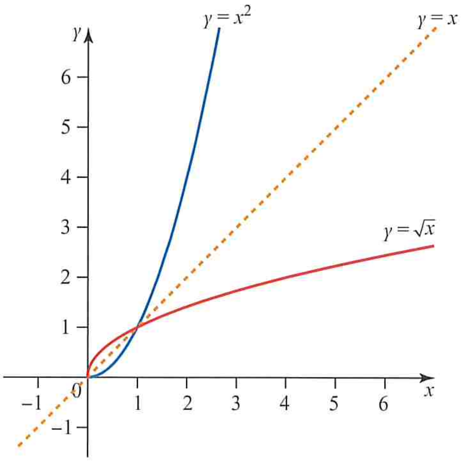


Figure 3.31

However, the function $y = x^2$ for all values of x does not have a single inverse, since $y = \pm\sqrt{x}$ is not a function. (A function must have a unique value of y for each value of x.)

Summary

To find the inverse of a function:

- write the function in the form $y = \mathrm{f}(x)$
- interchange x and y to give $x = \mathrm{f}(y)$
- rearrange to make y the subject.

Example 3.13

(i) Draw the graph of the function $\mathrm{f}(x) = x^2 - 2$ for $x \geqslant 0$ using the same scale on both axes.

(ii) Use algebra to work out the inverse function.

(iii) Using your graph from part (i) as a guide, sketch the graphs of $y = x$, $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same axes. Mark the points of intersection with the axes.

Solution

(i)

x	0	1	2	3
x^2	0	1	4	9
$\mathrm{f}(x) = x^2 - 2$	–2	–1	2	7

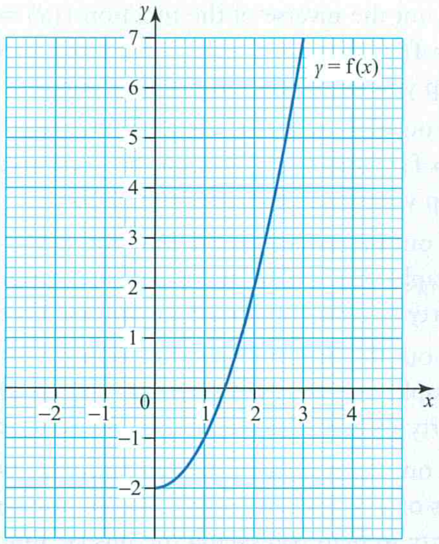


Figure 3.32

(ii) Write the function as $y = x^2 - 2$.

Interchange x and y to give $x = y^2 - 2$.

Make y the subject: $y^2 = x + 2$

$$y = \sqrt{x+2}.$$

(iii) The graph of $y = \mathrm{f}^{-1}(x)$ is the reflection of the graph of $y = \mathrm{f}(x)$ in the line $y = x$.

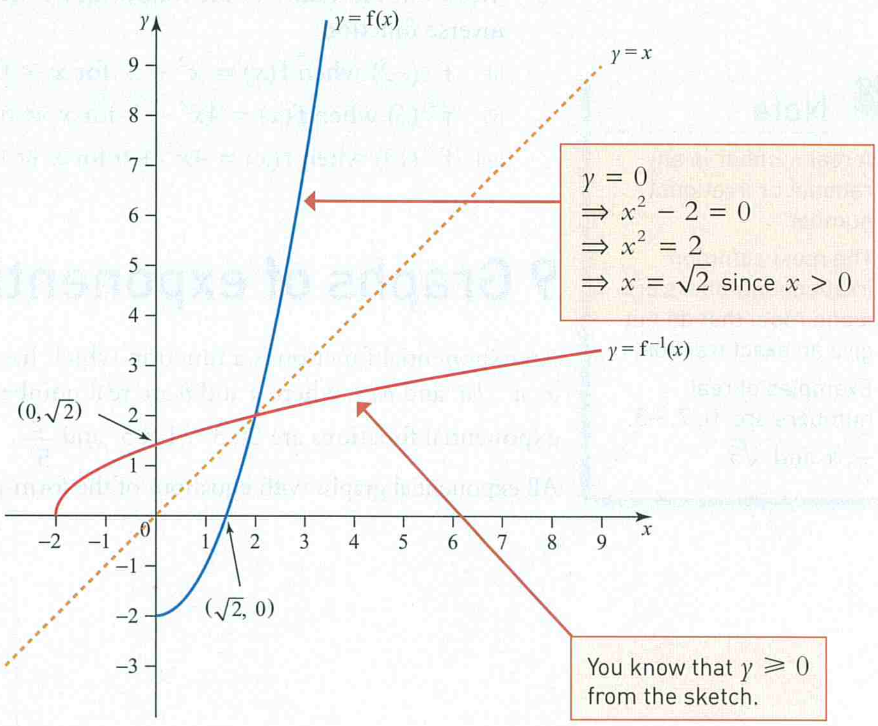


Figure 3.33

Discussion points

- ➜ In the Oxford English dictionary, the mathematical definition of 'inverse' is 'a reciprocal quantity' and the mathematical definition of 'reciprocal' is 'the quantity obtained by dividing the number one by the given quantity'.
- ➜ Comment on these definitions in the light of what you have just learnt.

Exercise 3G

① Work out the inverse of the function $f(x) = 2x - 3$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

② Work out the inverse of the function $f(x) = 3x - 2$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

③ Work out the inverse of the function $f(x) = x^2 - 4$ for $x \geqslant 0$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

④ Work out the inverse of the function $f(x) = (x - 2)^2$ for $x \geqslant 2$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

⑤ Work out the inverse of the function $f(x) = 2\sqrt{x}$ for $x \geqslant 0$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

⑥ Work out the inverse of the function $f(x) = \frac{1}{x}$ for $x > 0$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

⑦ Work out the values of the following by first finding an expression for the inverse function.

(i) $f^{-1}(4)$ when $f(x) = 5x - 1$

(ii) $f^{-1}(3)$ when $f(x) = 2 + \frac{1}{x}$ for $x > 0$

(iii) $f^{-1}(9)$ when $f(x) = x^2$ for $x \geqslant 0$

⑧ Work out the values of the following by first finding an expression for the inverse function.

(i) $f^{-1}(-2)$ when $f(x) = x^2 - 3$ for $x \geqslant 0$

(ii) $f^{-1}(3)$ when $f(x) = 4x^2 - 1$ for $x \geqslant 0$

(iii) $f^{-1}(13)$ when $f(x) = 4x^2 + 9$ for $x \geqslant 0$

9 Graphs of exponential functions

An exponential function is a function which has the variable as a power, such as a^x, a^{-x}, ba^x and ba^{-x}, where a and b are real numbers and $a > 0$. Some examples of exponential functions are 2^x, 3^{-x}, 4×5^x and $\frac{2}{5^x}$.

All exponential graphs with equations of the form $y = a^x$ pass through the point $(0, 1)$.

Note

A real number is any rational or irrational number.

The most common irrational numbers are π and roots that do not give an exact fraction.

Examples of real numbers are: $0, 7, -3, \frac{2}{3}, \pi$ and $\sqrt{5}$

Example 3.14

Plot the graphs of $y = 2^x$ and $y = 2^{-x}$ on the same axes for $-2 \leqslant x \leqslant 2$.

Solution

x	-2	-1	0	1	2
$y = 2^x$	0.25	0.5	1	2	4
$y = 2^{-x}$	4	2	1	0.5	0.25

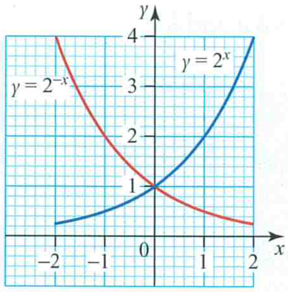


Figure 3.34

Note

Both of these graphs pass through the point (0, 1). This is true for all graphs of the form $y = a^x$ and $y = a^{-x}$, where a is a positive constant.

The effect of replacing x by $-x$ is to reflect the graph in the y-axis as shown in Figure 3.34.

Example 3.15

(i) Sketch, on the same axes, the graphs of $y = 2^x$, $y = 2 + x$ and $y = 2 - x$.

(ii) State the number of roots of these equations.

(a) $2^x = 2 + x$ (b) $2^x = 2 - x$

Solution

(i) $y = 2^x$ is as in the previous example.

$y = 2 + x$ is a straight line through (0, 2) with a gradient of 1.

$y = 2 - x$ is a straight line through (0, 2) with a gradient of (−1).

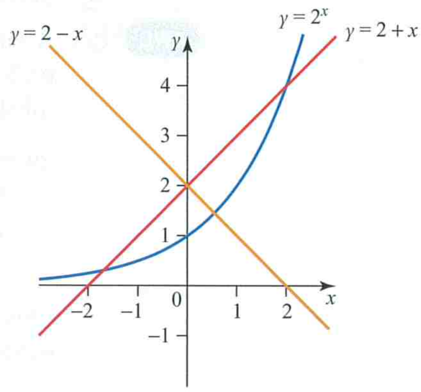


Figure 3.35

(ii) (a) There are two points of intersection of the graphs $y = 2^x$ and $y = 2 + x$, therefore there are two roots of the equation $2^x = 2 + x$.

(b) There is one point of intersection of the graphs $y = 2^x$ and $y = 2 - x$, therefore there is one root of the equation $2^x = 2 - x$.

Real life application of exponential functions (using differentiation, which is introduced in Chapter 8)

One important application of the exponential function is in the theory of radioactive decay. It has been shown that the rate of decay of a radioactive material is proportional to the amount of material present.

Example 3.16

The points (0, 0.5) and (3, 0.004) are on the curve

$y = ab^{-x}$.

Work out the values of a and b.

Solution

When $x = 0$, $y = 0.5$, so $0.5 = ab^0$

As $b^0 = 1$, $a = 0.5$

When $x = 3$, $y = 0.004$, so $0.004 = 0.5b^{-3}$

$0.008 = b^{-3}$

$125 = b^3$

$b = 5$

Exercise 3H

① Sketch the graphs of $y = 2^x$ and $y = 3^x$ on the same axes.

② Sketch the graphs of $y = 3^x$ and $y = 3^{-x}$ on the same axes.

③ Sketch the graphs of $y = 2^x$, $y = 2^x - 1$ and $y = 2^x + 1$ on the same axes.

④ Sketch the graphs of $y = 3^{-x}$, $y = 3^{-x} - 1$ and $y = 3^{-x} + 1$ on the same axes.

RWC ⑤ A virus is spreading among the inhabitants of a remote island and it will be 5 days before an antidote can be shipped there. Initially there are 100 inhabitants on the island. The number of people surviving after t hours is given by $N = 100 \times 10^{-\frac{t}{1000}}$.

(i) Calculate how many people are surviving after 24 hours.

(ii) Calculate how many people are surviving at the beginning of the fifth day when the vaccine is delivered.

Once the vaccine has been administered, it is a further 24 hours before it begins to be effective.

(iii) How many more people die before the vaccine begins to take effect?

⑥ Use any graphing software at your disposal to draw the graphs of $y = 2 \times 3^x$, $y = 3 \times 2^x$ and $y = 6^x$ on the same axes and write down what you notice.

⑦ (i) Use any graphing software at your disposal to draw the graphs of $y = 2 \times 4^x$, $y = 4 \times 2^x$ and $y = 8^x$ on the same axes.

(ii) In what ways is the result the same as in the previous question and in what ways is it different?

⑧ Use any graphing software at your disposal to draw the graphs of $y = 2^x + 2^{-x}$ and $y = 3^x + 3^{-x}$ on the same axes and write down what you notice.

10 Graphs of functions with up to three parts to their domains

A function may be defined with more than one part to its domain.

Here are two examples.

$$\begin{aligned} f(x) &= x + 1 \quad & -2 \leqslant x < 1 \\ &= 2 & 1 \leqslant x \leqslant 4 \end{aligned}$$

The domain of f(x) is $-2 \leqslant x \leqslant 4$.

$$\begin{aligned} g(x) &= 3 \quad & 0 \leqslant x < 2 \\ &= 7 - 2x & 2 \leqslant x < 5 \\ &= 3x + 12 & 5 \leqslant x \leqslant 6 \end{aligned}$$

The domain of g(x) is $0 \leqslant x \leqslant 6$.

Example 3.17

Draw the graph of $y = f(x)$ where

$$\begin{aligned} f(x) &= 4 \quad & -4 \leqslant x < -2 \\ &= x^2 & -2 \leqslant x < 2 \\ &= 8 - 2x & 2 \leqslant x \leqslant 4 \end{aligned}$$

Solution

The graph is drawn for each part of the given domain.

$y = 4$ is a horizontal line.

$y = x^2$ is a quadratic curve.

$y = 8 - 2x$ is a straight line with gradient -2.

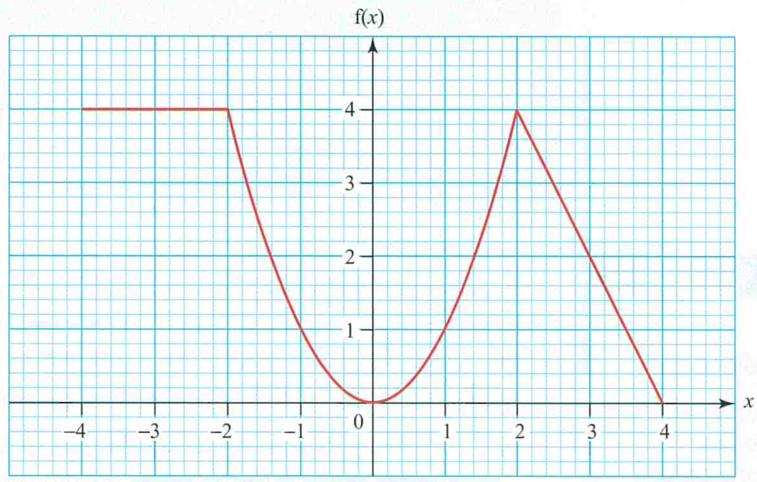


Figure 3.36

Example 3.18

Here is the graph of $y = \mathrm{f}(x)$.

(i) Define $\mathrm{f}(x)$, stating clearly the domain for each part.

(ii) State the range of $\mathrm{f}(x)$.

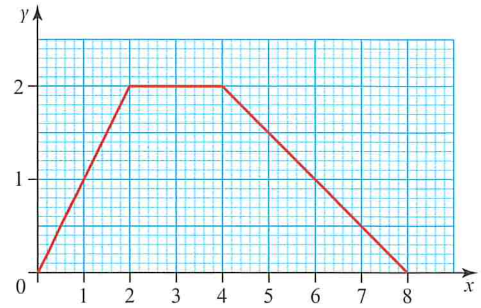


Figure 3.37

Solution

(i) For $0 \leqslant x < 2$ gradient $= \dfrac{2-0}{2-0}$

$= 1$

Line passes through (0, 0) so $y = x$

For $2 \leqslant x < 4$ horizontal line so $y = 2$

For $4 \leqslant x \leqslant 8$ gradient $= \dfrac{2-0}{4-8}$

$= \dfrac{2}{-4}$

$= -\dfrac{1}{2}$

Line passes through (8, 0) so $y - 0 = -\dfrac{1}{2}(x - 8)$

$y = -\dfrac{1}{2}x + 4$

$$\mathrm{f}(x) = x \qquad 0 \leqslant x < 2$$
$$= 2 \qquad 2 \leqslant x < 4$$
$$= -\tfrac{1}{2}x + 4 \qquad 4 \leqslant x \leqslant 8$$

> No x value should be included more than once in the domain.

(ii) The range is obtained by looking at the y values on the graph.

$0 \leqslant \mathrm{f}(x) \leqslant 2$

Example 3.19

> **Prior knowledge**
>
> You will have studied speed–time and distance–time graphs from GCSE. These provide a number of examples where there are up to three parts to their domain.

A car sets off from rest and accelerates uniformly for 5 seconds, after which it has reached a speed of $12\,\mathrm{m\,s^{-1}}$. After travelling for a further minute at that speed the driver notices the lights ahead change to red so decelerates uniformly for 5 seconds, coming to rest at the lights.

(i) Draw a graph to represent the journey, plotting time t seconds on the horizontal axis for $0 \leqslant t \leqslant 80$ and speed $v\,\mathrm{m\,s^{-1}}$ on the vertical axis for $0 \leqslant v \leqslant 15$.

(ii) Work out the acceleration for each of the three stages of the journey.

(iii) Work out the total distance travelled.

Solution

(i) While accelerating, the graph is represented by a straight line from (0, 0) to (5, 12). The part of the journey at constant speed is represented by a straight line from (5, 12) to (65, 12) since a minute is 60 seconds. Finally, the car comes to rest after a further 5 seconds, so join the point (65, 12) to (70, 0). Here is the graph of $v = f(t)$.

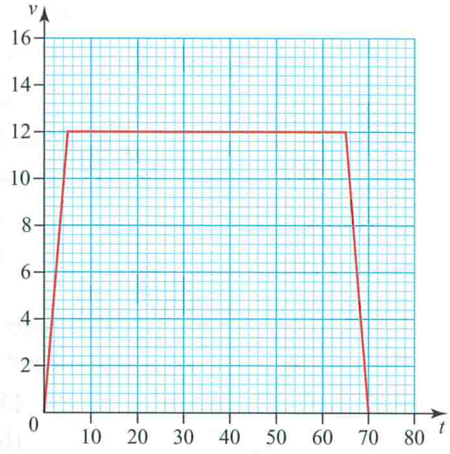


Figure 3.38

(ii) Acceleration = (change in speed) ÷ (change in time).

Stage 1: Acceleration $= \dfrac{12 - 0}{5 - 0} = 2.4\,\text{m s}^{-2}$

Stage 2: Constant speed so acceleration = 0

Stage 3: Acceleration $= \dfrac{0 - 12}{5 - 0} = -2.4\,\text{m s}^{-2}$

(iii) The distance travelled is represented by the area under a velocity–time graph.

In this case the shape of the graph is a trapezium, so

Area = (half the sum of parallel sides) × (distance between them)

$$= \left(\frac{60 + 70}{2}\right) \times 12 = 780$$

The distance travelled is 780 metres.

> Alternatively, the distance can be calculated as the sum of three separate areas:
> triangle + rectangle + triangle.

Exercise 3I

① Draw the graph of $y = f(x)$ where

$f(x) = 2 \qquad -2 \leqslant x < 1$
$\quad\; = 2x \qquad 1 \leqslant x \leqslant 3$

② Draw the graph of $y = f(x)$ where

$f(x) = x^2 \qquad 0 \leqslant x < 3$
$\quad\; = 9 \qquad 3 \leqslant x \leqslant 5$

③ Draw the graph of $y = g(x)$ where

$g(x) = x + 3 \qquad -3 \leqslant x < 0$
$\quad\; = 3 - x \qquad 0 \leqslant x \leqslant 3$

④ Draw the graph of $y = f(x)$ where

$f(x) = 3x - 1 \qquad -2 \leqslant x < 1$
$\quad\; = 3 - x \qquad 1 \leqslant x < 4$
$\quad\; = -1 \qquad 4 \leqslant x \leqslant 6$

⑤ Draw the graph of $y = f(x)$ where

$$f(x) = \begin{cases} 2x & -2 \leqslant x < 0 \\ \frac{1}{2}x & 0 \leqslant x < 2 \\ 5 - 2x & 2 \leqslant x \leqslant 4 \end{cases}$$

⑥ Draw the graph of $y = g(x)$ where

$$g(x) = \begin{cases} -4 & -3 \leqslant x < -2 \\ -x^2 & -2 \leqslant x < 2 \\ 3x - 10 & 2 \leqslant x \leqslant 4 \end{cases}$$

⑦ Figure 3.39 shows the graph of $y = f(x)$.

(i) Define $f(x)$, stating clearly the domain for each part.

(ii) State the range of $f(x)$.

(iii) Solve $f(x) = 5$.

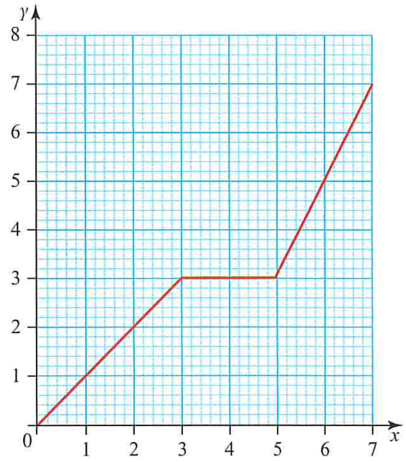


Figure 3.39

⑧ Figure 3.40 shows the graph of $y = f(x)$.

(i) Define $f(x)$, stating clearly the domain for each part.

(ii) State the range of $f(x)$.

(iii) Solve $f(x) = 3$.

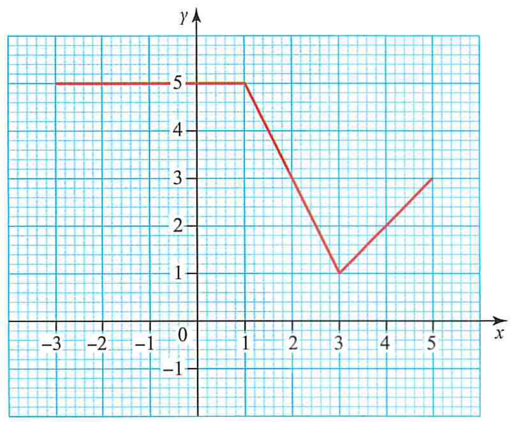


Figure 3.40

⑨ The graph of $y = f(x)$ is shown in Figure 3.41.

(i) Define $f(x)$, stating clearly the domain for each part.

(ii) Work out the area between the graph and the x-axis.

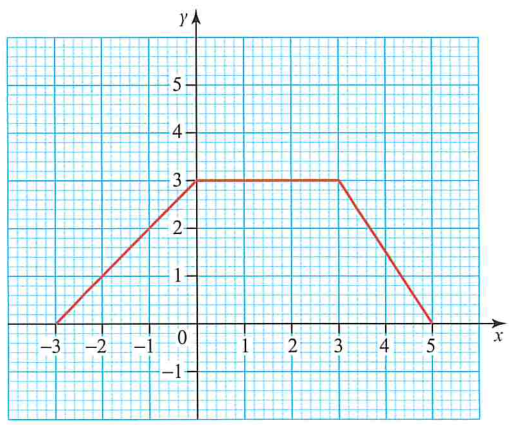


Figure 3.41

⑩ The graph of $y = g(x)$ is shown in Figure 3.42.

(i) Define $g(x)$, stating clearly the domain for each part.

(ii) Work out the area between the graph and the x-axis.

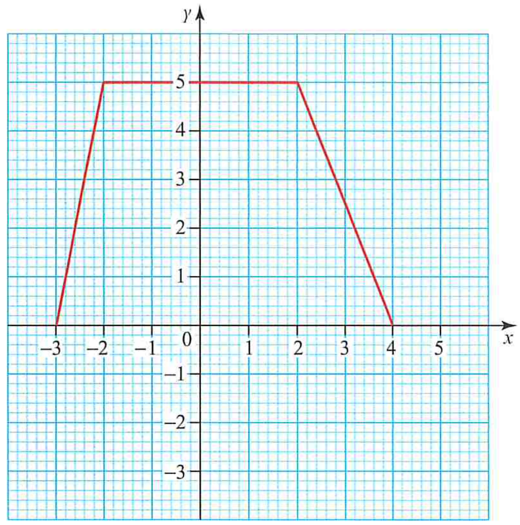


Figure 3.42

⑪ The distance–time graph in Figure 3.43 shows the relationship between distance travelled and time for a student who leaves home at 8:15 a.m., walks to the bus stop and then catches the bus to school.

(i) Describe what is happening between O and A.

(ii) Why do you think the distance doesn't change between A and B?

(iii) Think about the part of the graph from B to C. What does this tell you about the bus journey? How realistic is that?

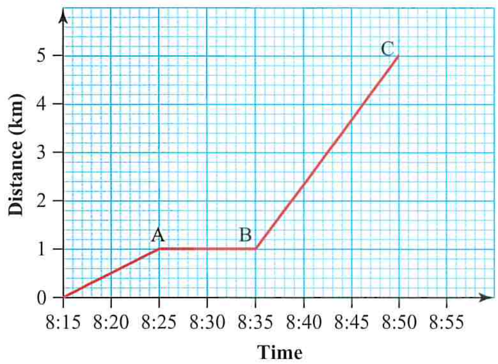


Figure 3.43

FUTURE USES

- You will use the techniques introduced here both in the next chapter and throughout your study of A-Level Pure Mathematics, but particularly in the section on graphs and their transformations.
- The section on graphs of inverse functions is not part of the Level 2 Further Mathematics specification, but you will meet it again at A-Level.
- Exponential functions also have applications in Mechanics and Physics in connection with problems on growth and decay.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- state the domain and range of a function on graph paper
- draw the graph of a function on graph paper
- sketch a graph – not using graph paper
- find the equation of a line given either the coordinates of two points on the line or the gradient of the line and the coordinates of one point
- recognise the shape of a quadratic graph from its equation, determining whether it has a maximum or minimum turning point
- recognise the graphs of the basic exponential functions $y = a^x$ and $y = a^{-x}$ and functions associated with them
- interpret graphs which relate to practical or physical situations.

KEY POINTS

1 A function maps an input, x, to an output, $\mathrm{f}(x)$.

2 The set of input values is the domain of $\mathrm{f}(x)$.
The set of output values is the range of $\mathrm{f}(x)$.

3 A composite function occurs when two or more functions act in succession.

4 When asked to draw a graph, use graph paper.

5 When asked to sketch a graph, do not use graph paper.

6 The gradient of the straight line joining the points (x_1, y_1) and (x_2, y_2) is given by $\dfrac{y_2 - y_1}{x_2 - x_1}$.

7 The equation of a straight line may take any of these forms.
- Line parallel to the y-axis: $x = a$
- Line parallel to the x-axis: $y = b$
- Line through the origin with gradient m: $y = mx$
- Line through $(0, c)$ with gradient m: $y = mx + c$
- Line through (x_1, y_1) with gradient m: $y - y_1 = m(x - x_1)$
- Line through (x_1, y_1) and (x_2, y_2):
$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \text{ or } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

8 The shape of a quadratic graph is a parabola. It may be either ∪ shaped (when the coefficient of the squared term is positive) or ∩ shaped (when it is negative).

9 For any one-to-one function $\mathrm{f}(x)$ there is an inverse function $\mathrm{f}^{-1}(x)$.

10 The inverse of a function 'undoes' the effect of the function.

11 The graphs of a function and its inverse are reflections of each other in the line $y = x$, provided that the same scale is used on both axes.

12 An exponential function is a function of the form $y = a^x$ or $y = a^{-x}$ where $a > 0$.

13 Functions may be defined with more than one part to their domain. In this case consider each part separately.

4 Algebra IV

'Obvious' is the most dangerous word in mathematics

E. T. Bell

1 Quadratic equations

Solving a quadratic equation by factorising

When solving an equation by factorisation it is essential that all non-zero terms are moved to one side of the equation, leaving zero on the other side. This is due to a unique property of zero: when the product of two (or more) numbers or expressions is zero, then at least one of the numbers or expressions must be zero. No other number has such a property.

Example 4.1 Solve $x^2 = 4x + 21$

Solution

$$x^2 = 4x + 21$$

$$\Rightarrow \quad x^2 - 4x - 21 = 0$$

$$\Rightarrow \quad (x + 3)(x - 7) = 0$$

$$\Rightarrow \quad x + 3 = 0 \quad \text{or} \quad x - 7 = 0$$

$$\Rightarrow \quad x = -3 \quad \text{or} \quad x = 7$$

Note

The *solution* of the equation is the *pair* of values $x = -3$ or $x = 7$.

The *roots* of the equation are the *individual* values $x = -3$ and $x = 7$.

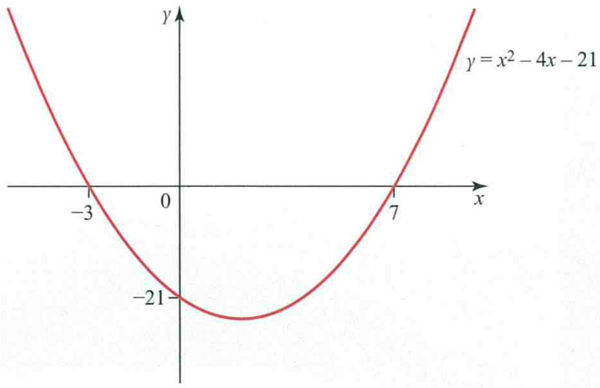


Figure 4.1

! Before solving a quadratic equation by factorisation, ensure that all non-zero terms are on one side of the equation only.

Example 4.2

Solve $8x^2 + 10x = 3$

Solution

First, move the 3 to the left-hand side of the equation, leaving zero on the other side.

$$8x^2 + 10x = 3$$
$$\Rightarrow \quad 8x^2 + 10x - 3 = 0$$
$$\Rightarrow \quad 8x^2 + 12x - 2x - 3 = 0$$
$$\Rightarrow \quad 4x(2x + 3) - 1(2x + 3) = 0$$
$$\Rightarrow \quad (2x + 3)(4x - 1) = 0$$
$$\Rightarrow \quad 2x + 3 = 0 \quad \text{or} \quad 4x - 1 = 0$$
$$\Rightarrow \quad x = -\frac{3}{2} \quad \text{or} \quad x = \frac{1}{4}$$

Find two numbers whose sum is 10 and product is −24 (10 is the coefficient of x, and −24 is the product of the constant term and the coefficient of x^2).

The numbers required are 12 and −2.

Sometimes a quadratic equation will not factorise.

In this case, you must complete the square or use the quadratic formula.

Solving a quadratic equation by completing the square

Example 4.3

Solve $x^2 - 8x + 3 = 0$

Solution

Subtract the constant term from both sides of the equation.

$$\Rightarrow \quad x^2 - 8x = -3$$

Add 16 to both sides of the equation $\Rightarrow \quad x^2 - 8x + 16 = -3 + 16$

Factorise the left-hand side $\Rightarrow \quad (x - 4)^2 = 13$

Square root both sides $\Rightarrow \quad x - 4 = \pm\sqrt{13}$

Add 4 to both sides $\Rightarrow \quad x = 4 + \sqrt{13}$ or $x = 4 - \sqrt{13}$

$$\Rightarrow \quad x = 7.6055\ldots \text{ or } x = 0.3944\ldots$$

Consider the coefficient of x (-8).
Halve it (-4).
Then square the half (16).
We then add this to both sides of the equation.
This process is called 'completing the square'.

If the square has been completed correctly, the left-hand side will always factorise to the form $(x \pm p)^2$.

If the coefficient of the squared term is not 1, then first divide the equation by the coefficient.

Example 4.4 Solve $2x^2 + 3x - 7 = 0$

Solution

$$
\begin{aligned}
2x^2 + 3x - 7 &= 0 \\
\Rightarrow \quad x^2 + \frac{3}{2}x - \frac{7}{2} &= 0 \\
\Rightarrow \quad x^2 + \frac{3}{2}x &= \frac{7}{2} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 &= \frac{7}{2} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 - \frac{9}{16} &= \frac{7}{2} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 &= \frac{56}{16} + \frac{9}{16} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 &= \frac{65}{16} \\
\Rightarrow \quad x + \frac{3}{4} &= \pm\sqrt{\frac{65}{16}} \\
\Rightarrow \quad x &= -\frac{3}{4} \pm \frac{\sqrt{65}}{4}
\end{aligned}
$$

Generalisation

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
\Rightarrow \quad x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} &= -\frac{c}{a} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\Rightarrow \quad x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
\Rightarrow \quad x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
\Rightarrow \quad x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Discussion points

- If $b^2 - 4ac$ is zero, how are the answers affected?
- If $b^2 - 4ac$ is negative, how are the answers affected?

The result $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as the *quadratic formula*. It can be used to solve any quadratic equation.

The $\pm$ sign indicates that there are two possible roots. One root is found by using the + sign, and the other by using the − sign.

Figure 4.2 shows a parabola – the shape of a quadratic curve.

The dotted line is the line of symmetry.

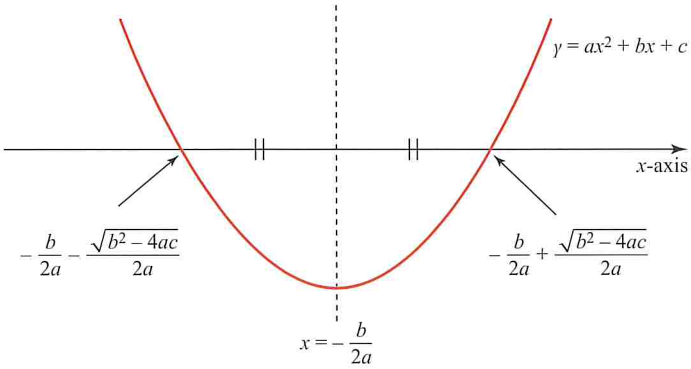


Figure 4.2

Solving a quadratic equation using the quadratic formula

Example 4.5

Use the quadratic formula to solve $3x^2 - 4x - 2 = 0$.

Solution

Comparing

$$3x^2 - 4x - 2 = 0$$

with

$$ax^2 + bx + c = 0$$

gives

$$a = 3, \quad b = -4, \quad c = -2$$

Using these values in the formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times -2}}{2 \times 3}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{6}$$

$$x = \frac{4 \pm \sqrt{40}}{6}$$

In a non-calculator paper, the answer could be simplified and left in an exact form as

$$x = \frac{2 \pm \sqrt{10}}{3}$$

Alternatively, in a calculator paper, approximate roots could be calculated as

$x = 1.72, \quad x = -0.39 \qquad$ (rounded to 2 d.p.)

Example 4.6

The length of a carpet is 1 m greater than its width. Its area is 9 m².

Work out the dimensions of the carpet to the nearest centimetre.

Solution

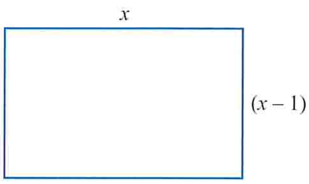


Figure 4.3

Let the length be x metres, so the width is $(x - 1)$ metres.

length × width = area

$$\text{so} \quad x(x-1) = 9$$

$$\Rightarrow \quad x^2 - x = 9$$

$$\Rightarrow \quad x^2 - x - 9 = 0 \text{ (collecting all terms on the left-hand side)}$$

Substituting $a = 1$, $b = -1$, $c = -9$ into the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{gives } x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-9)}}{2 \times 1}$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{37}}{2}$$

$$\Rightarrow \quad x = 3.541\ldots \text{ or } x = -2.541\ldots$$

Clearly a negative answer is not feasible, so the dimensions are

length = 3.54 m

width = 2.54 m (to the nearest cm).

Example 4.7

Solve

$$\frac{5}{a+1} - \frac{2a}{a^2-1} = \frac{1}{2}$$

Solution

First factorise $(a^2 - 1)$ as $(a + 1)(a - 1)$.

$$\frac{5}{a+1} - \frac{2a}{(a+1)(a-1)} = \frac{1}{2}$$

Multiply each term by $2(a + 1)(a - 1)$.

$$\Rightarrow \quad 2\cancel{(a+1)}^{1}(a-1) \times \frac{5}{\cancel{(a+1)}_{1}} - 2\cancel{(a+1)}^{1}\cancel{(a-1)}^{1} \times \frac{2a}{\cancel{(a+1)}_{1}\cancel{(a-1)}_{1}}$$

$$= \cancel{2}^{1}(a+1)(a-1) \times \frac{1}{\cancel{2}_{1}}$$

$$\Rightarrow \quad 10(a-1) - 4a = (a+1)(a-1)$$

$$\Rightarrow \quad 10a - 10 - 4a = a^2 - 1$$

$$\Rightarrow \quad 0 = a^2 - 6a + 9$$

$$\Rightarrow \quad 0 = (a-3)(a-3)$$

$$\Rightarrow \quad a = 3 \text{ (repeated root)}$$

Chapter 4 Algebra IV

Exercise 4A

① Solve the following equations by factorising.

(i) $x^2 - 8x + 12 = 0$ (ii) $m^2 - 4m + 4 = 0$

(iii) $p^2 - 2p - 15 = 0$ (iv) $a^2 + 11a + 18 = 0$

(v) $2x^2 + 5x + 2 = 0$ (vi) $4x^2 + 3x - 7 = 0$

(vii) $15t^2 + 2t - 1 = 0$ (viii) $24r^2 + 19r + 2 = 0$

(ix) $3x^2 + 8x = 3$ (x) $3p^2 = 14p - 8$

② Solve the following equations

(a) by completing the square

(b) by using the quadratic formula.

Give your answers correct to 2 decimal places.

(i) $x^2 - 2x - 10 = 0$ (ii) $x^2 + 3x - 6 = 0$

(iii) $x^2 + x - 8 = 0$ (iv) $2x^2 + x - 8 = 0$

(v) $2x^2 + 2x - 9 = 0$ (vi) $x^2 + x = 10$

(vii) $x^2 = 4x + 1$ (viii) $2x^2 - 8x + 5 = 0$

③ Solve the following equations by using the quadratic formula. Give your answers correct to 2 decimal places.

(i) $3x^2 + 5x = -1$ (ii) $4x^2 = -9x - 3$

(iii) $2x^2 + 11x = 4$ (iv) $4x^2 + 4 = 9x$

(v) $5x^2 + 1 = 10x$ (vi) $-9 - 11x = 3x^2$

④ The sides of a right-angled triangle, in centimetres, are x, $2x - 2$, and $x + 2$, where $x + 2$ is the hypotenuse. Use Pythagoras' theorem to work out their lengths.

PS ⑤ A rectangular lawn measures 8 m by 10 m and is surrounded by a path of uniform width x m. The total area of the path is $63\,\text{m}^2$. Work out the value of x.

PS ⑥ The difference between two positive numbers is 2 and the difference between their squares is 40. Work out the two numbers.

⑦ The formula $h = 15t - 5t^2$ gives the height h metres of a ball, t seconds after it is thrown up into the air.

(i) Work out the times when the height is 10 m.

(ii) After how long does the ball hit the ground?

⑧ The area of this triangle is $68\,\text{cm}^2$.

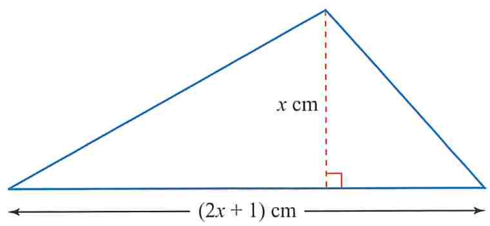


Figure 4.4

(i) Show that x satisfies the equation $2x^2 + x - 136 = 0$.

(ii) Solve the equation to work out the length of the base of the triangle.

9. Boxes are made by cutting squares of side 8 cm from the corners of rectangular sheets of cardboard and then folding the remaining card. The sheets of cardboard are 6 cm longer than they are wide.

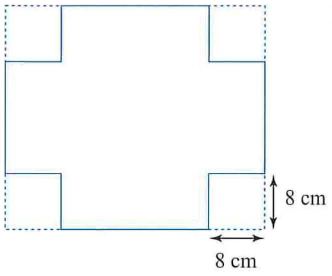


Figure 4.5

(i) For a sheet of cardboard with width x cm, write expressions, in terms of x, for
   (a) the length of the sheet
   (b) the length of the finished box
   (c) the width of the finished box.

(ii) Show that the volume of the box is $8x^2 - 208x + 1280$ cm³.

(iii) Work out the dimensions of the sheet of cardboard needed to make a box with a volume of 1728 cm³.

10. Solve the following equations.

(i) $\frac{1}{x} = 3 - \frac{2}{x+1}$ (ii) $\frac{2}{3x-1} + \frac{1}{x+8} = \frac{1}{2}$

(iii) $\frac{2}{a} - \frac{5}{2a-1} = 0$ (iv) $\frac{6}{p-2} + \frac{6}{p+1} = 1$

11. Solve the following equations.

(i) $\frac{1}{p} + p + 1 = \frac{13}{3}$

(ii) $1 + \frac{1}{x-1} = \frac{2x}{x+1}$

(iii) $\frac{6r}{r+1} - \frac{5}{r+3} = 3$

12. Solve the following equations.

(i) $\frac{a+4}{2a-3} = \frac{3(a+7)}{4(a+2)}$

(ii) $\frac{4x-13}{2x+1} = \frac{5x-23}{x+5}$

(iii) $\frac{2x+7}{x+7} = \frac{5x+13}{3-x}$

PS 13. A formula used in physics is $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ where f is the focal length of a mirror, u is the distance of the object from the mirror, and v is the distance of the image from the mirror. For a mirror with focal length 20 cm, work out the distance of the object from the mirror when the image is twice as far away from the mirror as is the object.

PS 14. Anna has used the quadratic formula to solve a quadratic equation.

She correctly calculated the answers as $x = \frac{4 \pm \sqrt{124}}{6}$.

Write down an equation that Anna might have solved.

2 Simultaneous equations in two unknowns

The equations you have met so far have only involved one unknown.

For example, $2x + 2 = x - 5$ or $a^2 - 3a + 2 = 0$

Figure 4.6 shows the line $x + y = 4$.

Discussion point

➜ When an equation involves two unknowns, for example $x + y = 4$, how many possible pairs of values are there for x and y?

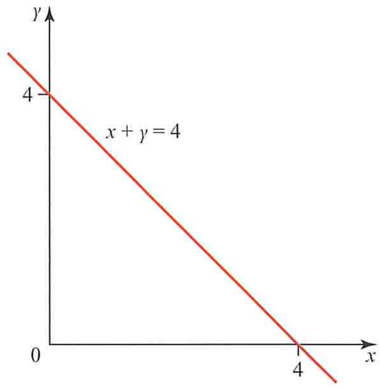


Figure 4.6

The coordinates of every point on that line give a pair of possible values for x and y. If the line $y = 2x + 1$ is included, as in Figure 4.7, the two lines can be seen to intersect at a single point.

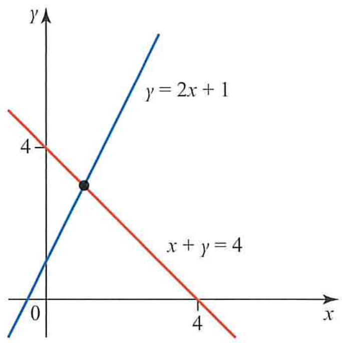


Figure 4.7

The coordinates of this point (1, 3) are the solution ($x = 1$, $y = 3$) to the *simultaneous equations*

$$x + y = 4$$

and $$y = 2x + 1.$$

There are several ways of solving simultaneous equations. You have just seen one method, that of drawing graphs. This is valid, but it has two drawbacks

(i) it is tedious

(ii) it may not be very accurate, particularly if the solution does not have integer values.

Solving simultaneous equations by substitution

Example 4.8

Solve the simultaneous equations

$$x + y = 4$$

$$y = 2x + 1$$

by substitution.

This method is particularly suitable when one of the unknowns is already written as the subject of one of the equations.

Solution

Take the expression for y from the second equation and substitute it into the first. This gives

$$x + (2x + 1) = 4$$

$$\Rightarrow \quad 3x = 3$$

$$\Rightarrow \quad x = 1$$

Then substitute $x = 1$ into one of the original equations, e.g. $y = 2x + 1$ giving $y = 2 \times 1 + 1 = 3$

So the solution is $x = 1$, $y = 3$ as indicated by the graphs.

Example 4.9

Figure 4.8 shows the graphs of $y = x^2 + x$ and $2x + y = 4$.

Solve the simultaneous equations

$$y = x^2 + x$$

and $\quad 2x + y = 4$

using the method of substitution.

This method is also suitable when one of the equations represents a curve.

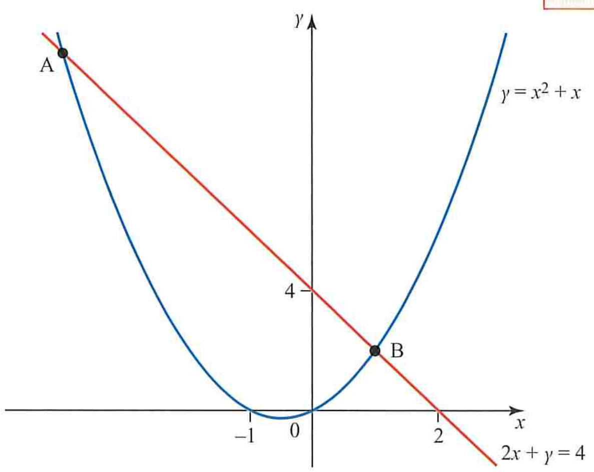


Figure 4.8

! Notice from Figure 4.8 that there are two points of intersection, A and B, so expect the solution to be two pairs of values for x and y.

Solution

$$y = x^2 + x \quad ①$$
$$2x + y = 4 \quad ②$$

Substitute for y from equation ① into equation ②.

$$2x + (x^2 + x) = 4$$
$$\Rightarrow \quad x^2 + 3x - 4 = 0$$
$$\Rightarrow \quad (x + 4)(x - 1) = 0$$
$$\Rightarrow \quad x = -4 \text{ or } x = 1$$

Substituting into $2x + y = 4$ ← Always substitute back into the linear equation.

$x = -4 \Rightarrow -8 + y = 4 \quad \Rightarrow y = 12$

$x = 1 \quad \Rightarrow 2 + y = 4 \quad \Rightarrow y = 2$

The solution is $x = -4$, $y = 12$ (point A) and $x = 1$, $y = 2$ (point B). ← Check your solution also fits equation ①.

Discussion point

→ Having found the values of x in the example, the values of y were found by substituting into the equation of the line. Why was it advisable to use the linear equation rather than the quadratic?

! The solution must always be given as pairs of values. It is wrong to write $x = -4$ or 1, $y = 12$ or 2, since not all pairs of values are possible.

Solving linear simultaneous equations by elimination

When both equations are linear and written in the same form, it may be preferable to use a process referred to as elimination.

Example 4.10

Solve the simultaneous equations

$$2x + y = 8 \quad ①$$
$$5x + 2y = 21 \quad ②$$

Solution

Notice that multiplying equation ① by 2 gives another equation containing $2y$.

$$5x + 2y = 21 \qquad \text{equation ②}$$
$$4x + 2y = 16 \qquad 2 \times \text{equation ①}$$

Subtracting $\Rightarrow \quad x \quad = 5$

Substitute $x = 5$ into equation ①

$$10 + y = 8 \quad \Rightarrow \quad y = -2$$

The solution is $x = 5$, $y = -2$

Sometimes you need to manipulate both equations to eliminate one of the unknowns, as in the following example.

Example 4.11

Solve the simultaneous equations

$$2x + 3y = -1 \quad (1)$$

$$3x - 2y = 18 \quad (2)$$

Solution

It is equally easy to eliminate x or y. It is up to you to choose which. The following method eliminates y.

$$4x + 6y = -2 \qquad 2 \times \text{equation } (1)$$

$$9x - 6y = 54 \qquad 3 \times \text{equation } (2)$$

Adding $\Rightarrow 13x = 52$

$\Rightarrow x = 4$

Substitute $x = 4$ into equation (1)

$8 + 3y = -1 \quad \Rightarrow \quad y = -3$

The solution is $x = 4$, $y = -3$

Discussion point

→ In Example 4.10 the equations were subtracted; in Example 4.11 they were added. How do you decide whether to add or subtract?

Simultaneous equations may arise in everyday problems.

Example 4.12

Tracey is buying fruit for a picnic.

Five apples and four pears cost exactly £2.20.

Two apples and six pears also cost exactly £2.20.

(i) Write this information as a pair of simultaneous equations.

(ii) Solve your equations to work out the cost of each type of fruit.

Solution

Let a pence be the cost of an apple and p pence be the cost of a pear.

> Make sure you introduce your unknowns.

> The cost of each piece of fruit will be a number of pence, so writing £2.20 as 220 pence avoids working with decimals.

(i) $5a + 4p = 220 \quad (1)$

$2a + 6p = 220 \quad (2)$

(ii) $\Rightarrow 15a + 12p = 660 \qquad 3 \times \text{equation } (1)$

$4a + 12p = 440 \qquad 2 \times \text{equation } (2)$

Subtracting $11a = 220$

$\Rightarrow a = 20$

Substitute $a = 20$ into equation (1)

$100 + 4p = 220$

$\Rightarrow p = 30$

An apple costs 20 pence and a pear costs 30 pence.

Example 4.13

A flag consists of a blue cross on a white background. Each white rectangle measures $2x$ cm by x cm, and the cross is y cm wide.

(i) Work out the total area of the flag in terms of x and y.

(ii) Show that the area of the cross is $6xy + y^2$.

(iii) The total area of the flag is 4500 cm^2 and the area of the cross is 1300 cm^2. Work out the values of x and y.

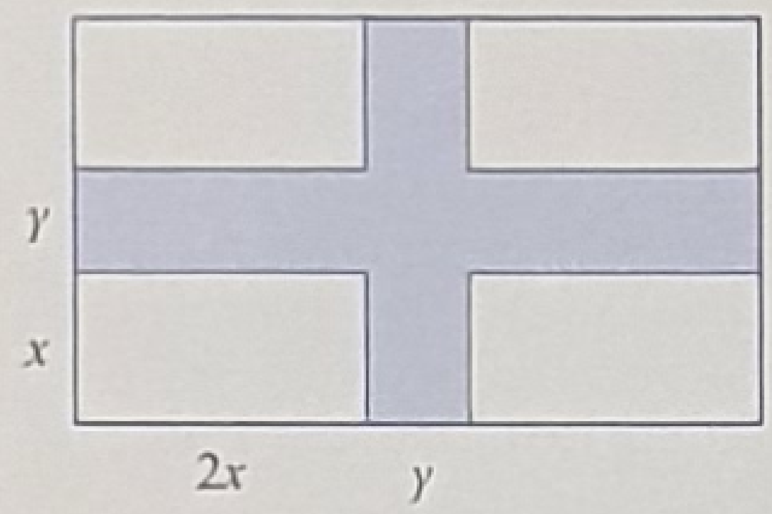


Figure 4.9

Solution

(i) length $= 4x + y$ and width $= 2x + y$

so $\quad$ area $= (4x + y)(2x + y)$

$$= 8x^2 + 6xy + y^2$$

(ii) Each white rectangle has an area of $2x \times x = 2x^2$.

$$\therefore \text{ area of cross} = 8x^2 + 6xy + y^2 - (4 \times 2x^2)$$

$$= 6xy + y^2$$

(iii)

$$8x^2 + 6xy + y^2 = 4500$$

$$6xy + y^2 = 1300$$

Subtracting $8x^2 \qquad = 3200$

$\Rightarrow \quad x^2 = 400$

$\Rightarrow \quad x = 20 \qquad$ (positive answer only)

Substitute $x = 20$ into $6xy + y^2 = 1300$

$$120y + y^2 = 1300$$

$\Rightarrow \quad y^2 + 120y - 1300 = 0$

$\Rightarrow \quad (y + 130)(y - 10) = 0$

$\Rightarrow \quad y = -130$ (reject since y is a length and so cannot be negative) or $y = 10$

$\Rightarrow \quad x = 20$ and $y = 10$

Exercise 4B

① Solve the following pairs of simultaneous equations using the substitution method.

(i) $y = x - 3$, $3x + 2y = 19$

(ii) $y = 2x - 9$, $4x - y = 17$

(iii) $y = 11 - 2x$, $2x + 5y = 37$

(iv) $y = 3x + 3$, $x - 2y = 4$

(v) $y = 7 - 2x$, $2x + 3y = 15$

(vi) $y = 3x - 5$, $x + 3y = -20$

② Solve the following pairs of simultaneous equations using the elimination method.

(i) $3x + 2y = 12$, $4x - y = 5$

(ii) $3x - 2y = 6$, $5x + 6y = 38$

(iii) $3x + 2y = 22$, $4x - 3y = 18$

(iv) $5x + 4y = 11$
$2x + 3y = 9$

(v) $4x + 5y = 33$
$3x + 2y = 16$

(vi) $4x - 3y = 2$
$5x - 7y = 9$

③ Solve the following pairs of simultaneous equations.

(i) $x + y = 5$
$x^2 + y^2 = 17$

(ii) $x - y + 1 = 0$
$3x^2 - 4y = 0$

(iii) $x^2 + xy = 8$
$x - y = 6$

(iv) $2x - y + 3 = 0$
$y^2 - 5x^2 = 20$

(v) $x = 2y$
$x^2 - y^2 + xy = 20$

(vi) $x + 2y = -3$
$x^2 - 2x + 3y^2 = 11$

PS ④ For each of the following situations, form a pair of simultaneous equations and solve them to answer the question.

(i) Three chews and four lollipops cost 72p. Five chews and two lollipops cost 64p. Work out the cost of a chew and the cost of a lollipop.

(ii) A taxi firm charges a fixed amount plus an extra fee per mile. A journey of five miles costs £5.00 and a journey of seven miles costs £6.60. How much does a journey of two miles cost?

(iii) Three packets of crisps and two packets of nuts cost £1.45. Two packets of crisps and five packets of nuts cost £2.25. How much does one packet of crisps and four packets of nuts cost?

(iv) Two adults and one child paid £37.50 to go to the theatre. The cost for one adult and three children was also £37.50. How much does it cost for two adults and five children?

PS ⑤ The diagram shows the circle $x^2 + y^2 = 25$ and the line $x + y = 7$. Work out the coordinates of A and B.

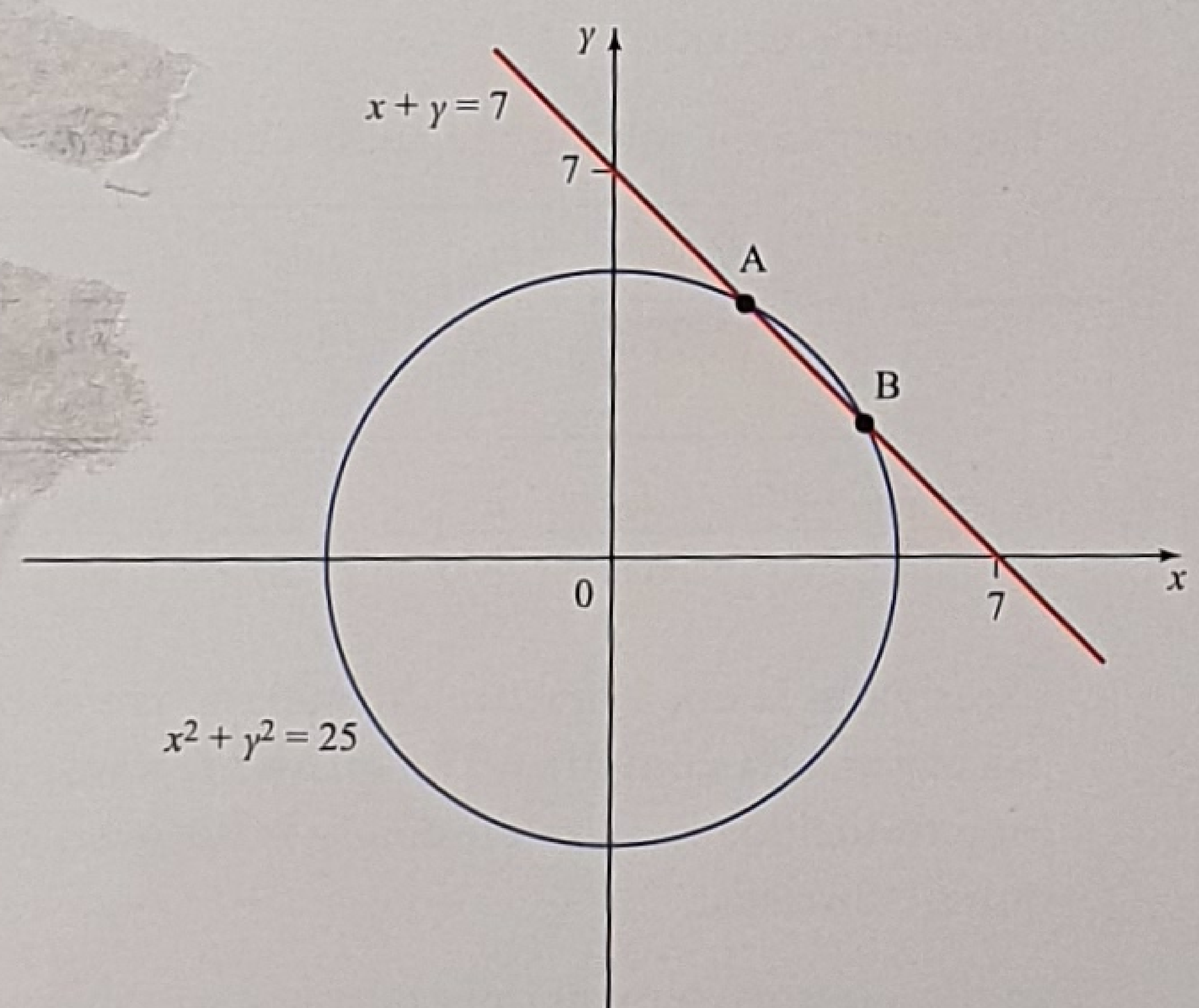


Figure 4.10

⑥ The sum of two numbers is 10. The product is –96

Work out the two numbers.

PS ⑦ (i) Work out the point of intersection of the circle $x^2 + y^2 = 8$ and the straight line $y - x = 4$

(ii) There is only one point of intersection of the circle and the line in part (i). Which of these diagrams is a sketch of the two graphs? Give a reason for your choice.

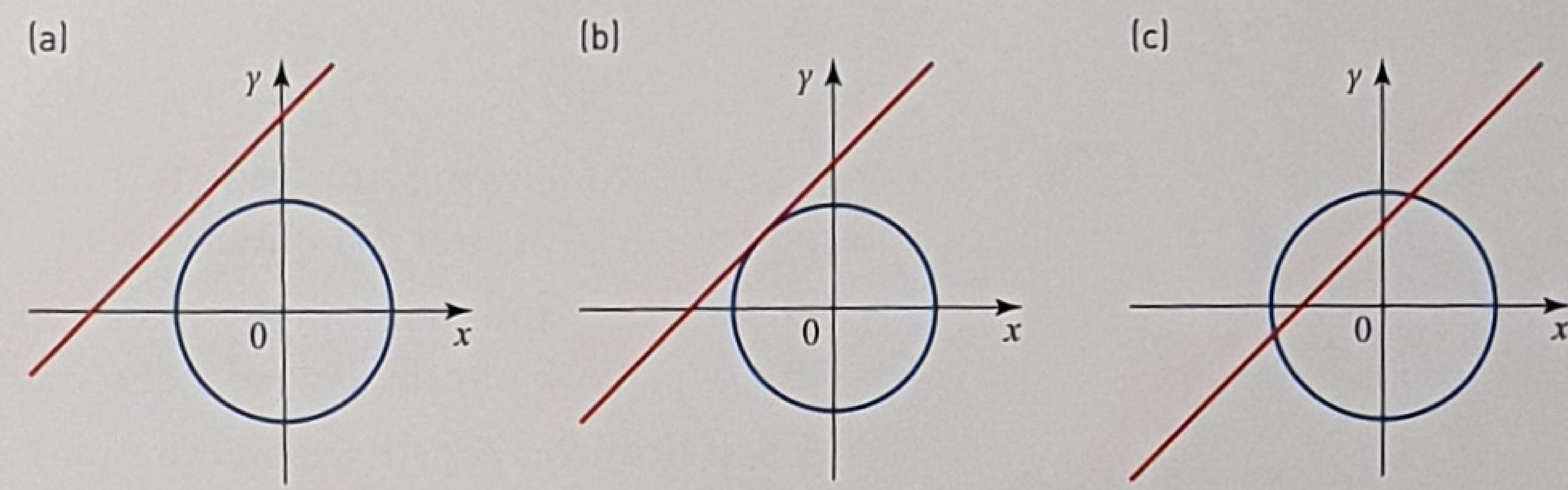


Figure 4.11

3 The factor theorem

Prior knowledge

Students should be familiar with function notation from their GCSE work.

The highest power in a quadratic is 2. Cubic expressions go up to 3, quartics to 4, quintics to 5, and so on. Such expressions are collectively referred to as polynomials. The degree of a polynomial is its highest power.

Note: a polynomial does not have negative or non-integer powers.

Just like the quadratic formula, there are formulae for solving cubic equations and quartic equations.

The formula for solving the cubic equation $ax^3 + bx^2 + cx + d = 0$ is

Students should not attempt to learn this formula. It is included here for interest only.

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

Clearly this is not a practical formula to use without a pre-programmed calculator, or a computer. The quartic formula is even more complicated, and long to include here. Interestingly, it has been proved to be impossible to w quintic formula.

In this course, you will only be asked to solve polyno be reduced to linear and/or quadratic factors.

FUTURE USES

- Some calculators can solve complicated equations. Suc use the Newton-Raphson method, which is a technique Mathematics at A-Level will learn. The above formula so the use of imaginary numbers even if the final answers a imaginary. Students of A-Level Further Mathematics will imaginary numbers (square roots of negative numbers).

Solving polynomial equations first involves use of the factor theorem.

Look at this quadratic equation.

$$x^2 - 5x - 6 = 0$$

Factorising $\Rightarrow$ $(x-6)(x+1) = 0$

$\Rightarrow$ $(x-6) = 0$ or $(x+1) = 0$

$\Rightarrow$ $x = 6$ or $x = -1$

Discussion points

- → What happens if you substitute $x = 6$ into $x^2 - 5x - 6$?
- → What about $x = -1$?

The factor theorem states this result in a general form.

If $(x - a)$ is a factor of the polynomial $f(x)$, then

- $f(a) = 0$
- $x = a$ is a root of the equation $f(x) = 0$.

Conversely, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

Example 4.14

Given that

$f(x) = x^3 + 2x^2 - x - 2$

(i) find $f(1)$, $f(-1)$, $f(2)$, $f(-2)$

(ii) and hence factorise $x^3 + 2x^2 - x - 2$

Solution

(i) $f(1) = 1 + 2 - 1 - 2$

$= 0$ $\Rightarrow$ $(x-1)$ is a factor

$f(-1) = (-1)^3 + 2(-1)^2 - (-1) - 2$

$= -1 + 2 + 1 - 2$

$= 0$ $\Rightarrow$ $(x+1)$ is a factor

$f(2) = 8 + 8 - 2 - 2$

$= 12$ $\Rightarrow$ $(x-2)$ is not a factor

$f(-2) = (-2)^3 + 2(-2)^2 - (-2) - 2$

$= -8 + 8 + 2 - 2$

$= 0$ $\Rightarrow$ $(x+2)$ is a factor

(ii) Hence $x^3 + 2x^2 - x - 2 = k(x-1)(x+1)(x+2)$
where k is a constant.
The coefficient of x^3 is 1, so k must be 1.
$f(x) = (x-1)(x+1)(x+2)$

Example 4.15

Given that

$f(x) = x^3 + 3x^2 - x - 3$

(i) show that $(x + 1)$ is a factor of $f(x)$

(ii) suggest other values of x you should try when looking for another factor

(iii) solve the equation $f(x) = 0$

Solution

(i) $f(-1) = (-1)^3 + 3(-1)^2 - (-1) - 3$

$= -1 + 3 + 1 - 3$

$= 0$

$\therefore (x + 1)$ is a factor of $f(x)$

(ii) Any other linear factor will be of the form $(x - a)$ where a is a factor of the constant term (-3).

This means that the only other values of x which are worth trying are 1, 3 and -3.

(iii) $f(1) = 1 + 3 - 1 - 3$

$= 0 \quad \Rightarrow \quad (x - 1)$ is a factor

$f(3) = 27 + 27 - 3 - 3$

$= 48$

$f(-3) = -27 + 27 + 3 - 3$

$= 0 \quad \Rightarrow \quad (x + 3)$ is a factor

As $f(x)$ is a cubic, then there are no more than three roots.

$x = -1, x = 1, x = -3$

Sometimes you may only be able to find one linear factor for the cubic and, in this case, you then need to use long division.

Example 4.16

Given that

$f(x) = x^3 - x^2 - 3x - 1$

(i) show that $(x + 1)$ is a factor

(ii) factorise $f(x)$

(iii) solve $f(x) = 0$

Discussion points

- What happens when you try $x = 1$?
- Is there any other value you should try?

Solution

(i) $f(-1) = (-1)^3 - (-1)^2 - 3(-1) - 1$

$= -1 - 1 + 3 - 1$

$= 0$

$\Rightarrow \quad (x + 1)$ is a factor of $x^3 - x^2 - 3x - 1$

(ii) Since $(x + 1)$ is a factor, then divide $f(x)$ by $(x + 1)$.

$$\begin{array}{r} x^2 - 2x - 1 \\ x+1 \overline{) x^3 - x^2 - 3x - 1} \\ \underline{x^3 + x^2} \quad\quad\quad \\ -2x^2 - 3x \quad \\ \underline{-2x^2 - 2x} \quad \\ -x - 1 \\ \underline{-x - 1} \\ 0 \end{array}$$

$x^3 + x^2$ is $x^2 \times (x + 1)$

$-2x^2 - 2x$ is $-2x \times (x + 1)$

$-x - 1$ is $-1 \times (x + 1)$

$x^2 - 2x - 1$ cannot be factorised, so $f(x)$ is now fully factorised.

So $f(x) = (x + 1)(x^2 - 2x - 1)$

(iii) $f(x) = 0 \Rightarrow (x + 1)(x^2 - 2x - 1) = 0$

$\Rightarrow$ either $x = -1$ or $x^2 - 2x - 1 = 0$

Using the quadratic formula on $x^2 - 2x - 1 = 0$

gives

$$x = \frac{2 \pm \sqrt{4 - (4 \times 1 \times (-1))}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= 2.414 \text{ or } -0.414$$

The complete solution is $x = -1, -0.414$ or 2.414 (to 3 d.p.)

! When using long division it is advisable to keep the terms in columns. This may mean that an extra zero term should be included to help with this.

Example 4.17

Given that $(x + 2)$ is a factor of $x^3 - 5x - 2$, work out a quadratic factor.

Solution

Method 1

$$\begin{array}{r} x^2 - 2x - 1 \\ x + 2 \overline{) x^3 + 0x^2 - 5x - 2} \\ \underline{x^3 + 2x^2} \qquad\qquad \\ -2x^2 - 5x \qquad \\ \underline{-2x^2 - 4x} \qquad \\ -x - 2 \\ \underline{-x - 2} \\ 0 \end{array}$$

$\Rightarrow$ $x^2 - 2x - 1$ is also a factor.

Method 2

As $(x + 2)$ is a factor, $x^3 - 5x - 2 \equiv (x + 2)(ax^2 + bx + c)$

You could compare coefficients as shown in Chapter 2.

However, it is clear that a must be 1 and c must be -1, so it is simpler to immediately write:

$x^3 - 5x - 2 \equiv (x + 2)(x^2 + bx - 1)$

and then compare coefficients of x^2 or x to find b:

$0 = 2 + b$ or $-5 = 2b - 1$ both of which give $b = -2$,

so $x^3 - 5x - 2 \equiv (x + 2)(x^2 - 2x - 1)$

An extension to the factor theorem includes simplified fractional roots.

$$f\left(\frac{b}{a}\right) = 0 \quad \Leftrightarrow \quad (ax - b) \text{ is a factor of } f(x)$$

Example 4.18

(i) Show that $(x + 1)$ and $(3x - 2)$ are factors of

$3x^4 + 4x^3 - 16x^2 - 7x + 10$

(ii) Hence solve $3x^4 + 4x^3 - 16x^2 - 7x + 10 = 0$

Solution

(i) Let $f(x) = 3x^4 + 4x^3 - 16x^2 - 7x + 10$

$$f(-1) = 3 \times (-1)^4 + 4 \times (-1)^3 - 16 \times (-1)^2 - 7 \times (-1) + 10$$

$$= 3 - 4 - 16 + 7 + 10$$

$$= 0$$

$\Rightarrow \quad (x + 1)$ is a factor of $3x^4 + 4x^3 - 16x^2 - 7x + 10$

$$f\left(\frac{2}{3}\right) = 3 \times \left(\frac{2}{3}\right)^4 + 4 \times \left(\frac{2}{3}\right)^3 - 16 \times \left(\frac{2}{3}\right)^2 - 7 \times \frac{2}{3} + 10$$

$$= \frac{16}{27} + \frac{32}{27} - \frac{64}{9} - \frac{14}{3} + 10$$

$$= 0$$

$\Rightarrow \quad (3x - 2)$ is also a factor of $3x^4 + 4x^3 - 16x^2 - 7x + 10$

(ii) $(x + 1)(3x - 2) = 3x^2 + x - 2$

$$\begin{array}{r}
x^2 + x - 5 \\
3x^2 + x - 2 \enclose{longdiv}{3x^4 + 4x^3 - 16x^2 - 7x + 10} \\
\underline{3x^4 + x^3 - 2x^2} \qquad\qquad \\
3x^3 - 14x^2 - 7x \qquad \\
\underline{3x^3 + x^2 - 2x} \qquad \\
-15x^2 - 5x + 10 \\
\underline{-15x^2 - 5x + 10} \\
0
\end{array}$$

$\Rightarrow \quad f(x) = (x + 1)(3x - 2)(x^2 + x - 5)$

This equals zero when

$$x + 1 = 0, \; 3x - 2 = 0, \; x^2 + x - 5 = 0$$

$$\Rightarrow x = -1 \text{ or } x = \frac{2}{3} \quad \text{or } x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -5}}{2 \times 1}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{21}}{2}$$

Exercise 4C

① Determine whether the following linear functions are factors of the given polynomials.

(i) $x^3 - 8x + 7$ $(x - 1)$

(ii) $x^3 + x^2 - 4x - 5$ $(x + 2)$

(iii) $x^4 - 6x^2 + 10x - 12$ $(x - 2)$

(iv) $x^5 + 32$ $(x + 2)$

(v) $2x^4 - x^3 - 20$ $(x + 2)$

(vi) $x^3 - ax^2 + a^2x - a^3$ $(x - a)$

Exam questions are unlikely to require students to find five different linear factors of an expression. However, students will be expected to feel comfortable working with polynomials of such a high degree.

② Factorise the following functions as a product of linear factors.

(i) $x^3 - 3x^2 - x + 3$ (ii) $x^3 - 7x - 6$

(iii) $x^3 - x^2 - 2x$ (iv) $x^3 - 2x^2 - 13x - 10$

(v) $x^3 - x^2 - 14x + 24$ (vi) $x^4 - 3x^3 - 11x^2 + 3x + 10$

(vii) $x^4 - 4x^3 + 6x^2 - 4x + 1$ (viii) $x^4 - 13x^2 + 36$

(ix) $x^5 - 4x^4 - 17x^3 + 24x^2 + 36x$ (x) $x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120$

③ Solve the following equations.

(i) $x^3 - 2x^2 - 5x + 6 = 0$ (ii) $x^3 + 3x^2 - 6x - 8 = 0$

(iii) $x^3 - 2x^2 - 21x - 18 = 0$ (iv) $x^4 + 3x^3 - 5x^2 - 3x + 4 = 0$

(v) $2x^3 + x^2 - 7x + 4 = 0$ (vi) $x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120 = 0$

④ $f(x) = x^3 + 2x^2 + ax - 76$

Given that $(x - 4)$ is a factor of $f(x)$, work out the value of a.

⑤ $f(x) = x^3 + px^2 + qx + 6$

(i) Given that $(x - 1)$ is a factor of $f(x)$, write an equation in p and q.

(ii) Given also that $(x + 3)$ is a factor of $f(x)$, write another equation in p and q.

(iii) Solve your simultaneous equations to work out the values of p and q.

PS ⑥ (i) Work out the value of k for which $x = 2$ is a root of $x^3 + kx + 6 = 0$

(ii) Work out the other roots when k takes this value.

PS ⑦ The diagram shows an open cuboid tank whose base is a square of side x metres and whose volume is 8 m^3.

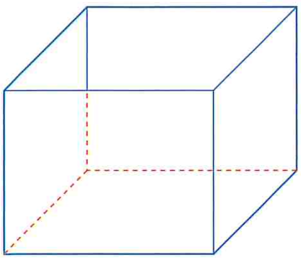


Figure 4.12

(i) Write down an expression in terms of x for the height of the tank.

(ii) Show that the surface area of the tank is $\left(x^2 + \dfrac{32}{x}\right)\text{ m}^2$.

(iii) Given that the surface area is 24 m^2, show that

$x^3 - 24x + 32 = 0$

(iv) Solve $x^3 - 24x + 32 = 0$ to work out the possible values of x.

PS ⑧ $(x - 1)$ and $(5x + 2)$ are factors of $5x^4 + px^3 - 10x^2 + qx + 2$

Solve $5x^4 + px^3 - 10x^2 + qx + 2 = 0$

4 Linear inequalities

Discussion point

- The radius of the Earth's orbit around the Sun is approximately 1.5×10^8 km; that of Mars is about 2.3×10^8 km. The Earth takes 365 days for one orbit and Mars takes 687 days.
- Given that the distance from Earth to Mars is x km, what can you say about x?

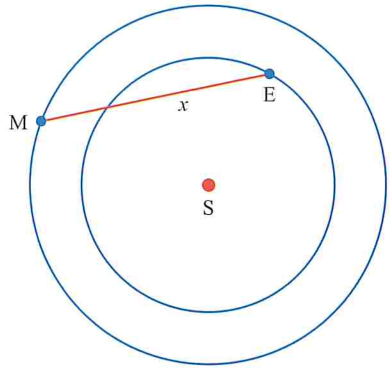


Figure 4.13

Discussion point

- What is the difference between an equation and an inequality?

Some inequalities appear as a combination of one, or more, expressions which must be solved in order to work out a range of possible values for the variable.

Example 4.19

Solve $2y + 6 < 5y + 12$

Discussion points

- In what ways is solving an inequality like solving an equation?
- In what ways is it different?
- Explain, with examples, why you need to reverse the inequality sign when you multiply or divide an inequality by a negative number.

Solution

Method 1

		$2y + 6 < 5y + 12$
Subtract $2y$	$\Rightarrow$	$6 < 3y + 12$
Subtract 12	$\Rightarrow$	$-6 < 3y$
Divide by 3	$\Rightarrow$	$-2 < y$
Make y the subject	$\Rightarrow$	$y > -2$

Method 2

		$2y + 6 < 5y + 12$
Subtract $5y$	$\Rightarrow$	$-3y + 6 < 12$
Subtract 6	$\Rightarrow$	$-3y < 6$
Divide by (-3)	$\Rightarrow$	$y > -2$

Example 4.20

Solve the inequality $5 < 3x - 1 \leqslant 17$

Solution

Add 1 throughout	$\Rightarrow$	$6 < 3x \leqslant 18$
Divide by 3	$\Rightarrow$	$2 < x \leqslant 6$

The following example, although algebraic, is solved using knowledge of number operations and number facts.

Example 4.21

(i) Given that $-2 < x < 5$, work out an inequality for x^2.

(ii) Given that $1 \leqslant a \leqslant 4$ and $-3 \leqslant b \leqslant 2$, work out an inequality for $a - b$.

Solution

(i) Squaring a negative number results in a positive number. Also $(-2)^2 < 5^2$.

So $0 \leqslant x^2 < 25$

(ii) The least value of $a - b$ will occur when a takes its least value and b takes its greatest value.

Least value $= 1 - 2$

$= -1$

The greatest value of $a - b$ will occur when a takes its greatest value and b takes its least value.

Greatest value $= 4 - (-3)$

$= 4 + 3$

$= 7$

So, $-1 \leqslant a - b \leqslant 7$

Exercise 4D

① Solve the following inequalities.

(i) $2x - 3 < 7$ (ii) $5 + 3x \geqslant 11$

(iii) $6y + 1 \leqslant 4y + 9$ (iv) $y - 4 > 3y - 12$

(v) $4x + 1 \geqslant 3x - 2$ (vi) $b - 3 \leqslant 5b + 9$

(vii) $\frac{x + 5}{2} > 1$ (viii) $\frac{2x - 3}{3} < 7$

(ix) $\frac{5 - 3x}{4} \leqslant 5$ (x) $\frac{2 - 4x}{3} \geqslant 6$

(xi) $4 \leqslant 5x - 6 \leqslant 14$ (xii) $11 \leqslant 3x + 5 \leqslant 20$

(xiii) $5 < 7 - 2x < 13$ (xiv) $5 > 9 - 4x > 1$

② Given that $0 \leqslant p \leqslant 3$ and $2 \leqslant q \leqslant 5$, work out the inequality for $p - q$.

③ Given that $-2 < x < 4$ and $1 < y < 3$, work out the inequality for $x + y$.

④ Given that $1 \leqslant a \leqslant 6$ and $-3 \leqslant b \leqslant 3$, work out inequalities for

(i) $a + b$ (ii) $a - b$.

⑤ Given that $-3 \leqslant a \leqslant 0$ and $-1 \leqslant b \leqslant 10$, work out inequalities for

(i) $a + b$ (ii) $a - b$ (iii) $2a + 3b$.

PS ⑥ Given that $x > 2$ and $y < 0$, decide whether each of the following statements are ALWAYS TRUE, SOMETIMES TRUE or NEVER TRUE.

(i) $4x > 8$ (ii) $2y > 0$ (iii) $x + y < 2$

(iv) $x^2 > 10$ (v) $y^2 < 0$ (vi) $x - y > 2$

PS ⑦ Given that $0 < x < 1$ and $y > 0$, decide whether each of the following statements are ALWAYS TRUE, SOMETIMES TRUE or NEVER TRUE.

(i) $\frac{1}{x} > 1$ (ii) $\frac{y}{x} > 0$ (iii) $x + y < 0$

(iv) $xy > 4$ (v) $x^2 > 1$ (vi) $x - y < 0$

PS ⑧ The square and rectangle have dimensions in centimetres.

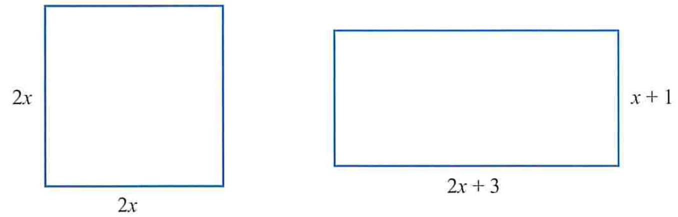


Figure 4.14

The perimeter of the square is greater than the perimeter of the rectangle. Which values of x satisfy this condition?

5 Quadratic inequalities

The quadratic inequalities in this section all involve quadratic expressions that factorise. This means that you can either find a solution by sketching the appropriate graph, or you can use line segments to reduce the quadratic inequality to two simultaneous linear inequalities.

Example 4.22

Solve

(i) $x^2 - 2x - 3 < 0$

(ii) $x^2 - 2x - 3 \geqslant 0$

Solution

Method 1

$x^2 - 2x - 3 = (x + 1)(x - 3)$

So the graph of $y = x^2 - 2x - 3$ crosses the x-axis when $x = -1$ and $x = 3$.

Look at the two graphs in Figure 4.15.

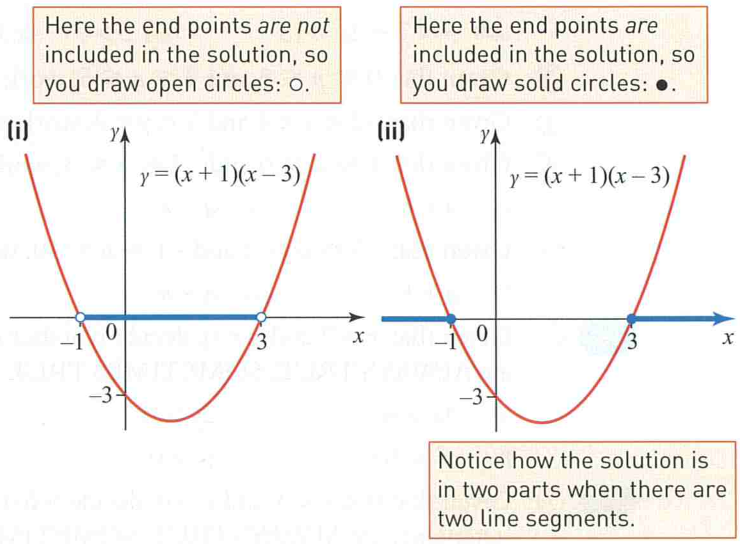


Figure 4.15

(i) You want the values of x for which $y < 0$, that is where the curve is below the x-axis. The solution is $-1 < x < 3$

(ii) You want the values of x for which $y \geqslant 0$, that is where the curve crosses or is above the x-axis. The solution is $x \leqslant -1$ or $x \geqslant 3$

An alternative method identifies the values of x for which each of the factors is zero and considers the sign of each factor in the intervals between these critical values.

Method 2

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
sign of $(x+1)$	$-$	0	$+$	$+$	$+$
sign of $(x-3)$	$-$	$-$	$-$	0	$+$
sign of $(x+1)(x-3)$	$(-)\times(-) = +$	$(0)\times(-) = 0$	$(+)\times(-) = -$	$(+)\times(0) = 0$	$(+)\times(+) = +$

Short method for solving $ax^2 + bx + c < 0$ (or $>$ or $\leqslant$ or $\geqslant$)

Step 1: Solve $ax^2 + bx + c = 0$ to get the critical values p and q ($p \leqslant q$)

Step 2: If $a < 0$ then multiply throughout by -1

Step 3: If the inequality is $<$ (or $\leqslant$) the solution is $p < x < q$ (or $p \leqslant x \leqslant q$)

Step 4: If the inequality is $>$ (or $\geqslant$) the solution is $x < p$ or $x > q$ (or $x \leqslant p$ or $x \geqslant q$)

! Don't forget to reverse the sign if you multiply (or divide) throughout any inequality by a negative value.

Example 4.23

Solve $2x + x^2 > 3$

Solution

$$2x + x^2 > 3$$

$$\Rightarrow \quad x^2 + 2x - 3 > 0$$

$$\Rightarrow \quad (x-1)(x+3) > 0$$

$\Rightarrow$ the critical values are -3 and 1

$\therefore$ the solution is $x < -3$ or $x > 1$

Select the outer regions because the inequality sign is >.

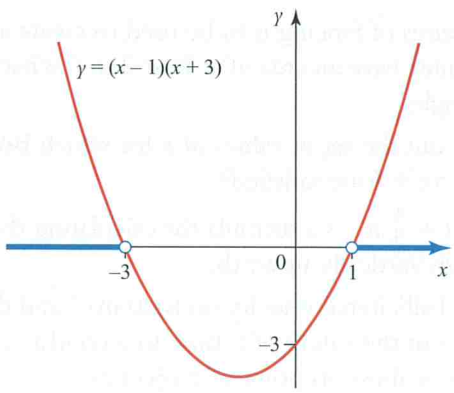


Figure 4.16

Exercise 4E

① Solve the following inequalities.

(i) $x^2 - 6x + 5 > 0$

(ii) $a^2 + 3a - 4 \leqslant 0$

(iii) $2y^2 + y - 3 < 0$

(iv) $4 - y^2 \geqslant 0$

(v) $x^2 - 4x + 4 > 0$

(vi) $p^2 - 3p \leqslant -2$

(vii) $(a + 2)(a - 1) > 4$

(viii) $8 - 2a \geqslant a^2$

(ix) $3y^2 + 2y - 1 > 0$

(x) $y^2 \geqslant 4y + 5$

② For which values of x are the following graphs below the x-axis?

(i) $y = x^2 - x - 6$

(ii) $y = x^2 + 2x - 8$

(iii) $y = x^2 + 6x + 8$

(iv) $y = x^2 - 5x + 6$

(v) $y = 2x^2 - x - 1$

(vi) $y = -x^2 - x + 6$

(vii) $y = 21 + 4x - x^2$

(viii) $y = 5x + 2 - 3x^2$

PS ③ The square and rectangle have dimensions in centimetres.

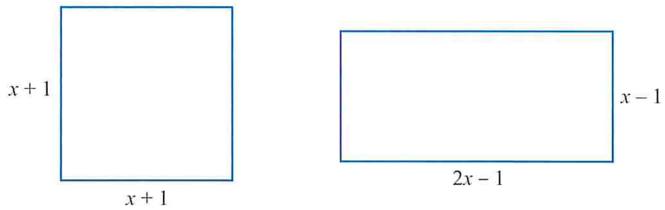


Figure 4.17

The area of the square is less than the area of the rectangle.

For which values of x is this condition satisfied?

PS ④ The triangle and parallelogram have dimensions in metres.

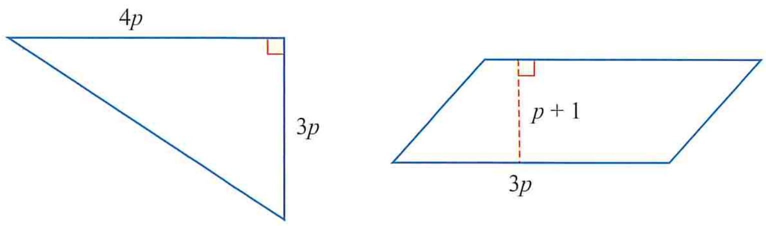


Figure 4.18

The area of the triangle is less than the area of the parallelogram.

For which values of p is this condition satisfied?

PS ⑤ 20 metres of fencing is to be used to create a rectangular sheep pen. If the pen must have an area of at least $21\,\text{m}^2$, what are the possible lengths of the rectangle?

PS ⑥ Work out the set of values of x for which **both** $2(x + 4) \geqslant 13$ **and** $x^2 < 5x + 6$ are satisfied?

PS ⑦ $s = ut + \frac{1}{2}at^2$ is a formula for calculating the height (s) of a ball when thrown vertically upwards.

If the ball's initial velocity (u) is $20\,\text{m s}^{-1}$ and the acceleration (a) is $-10\,\text{m s}^{-2}$, work out the values of t (time in seconds) for which the ball is more than 18.75 m above its point of projection.

PS ⑧ The graphs of $y = 4 - x^2$ and $y = x^2 - 2x - 3$ have been plotted on the same axes.

For which values of x is the graph of $y = 4 - x^2$ above the graph of $y = x^2 - 2x - 3$?

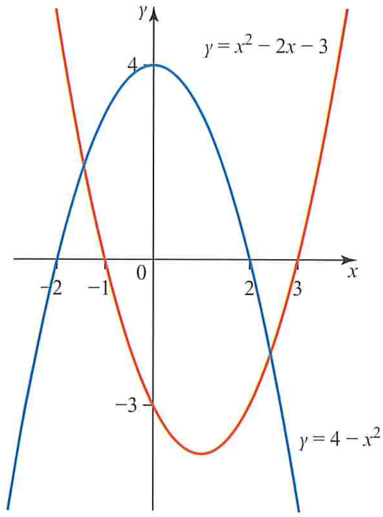


Figure 4.19

6 Indices

The three index laws are summarised here.

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{m \times n}$$

These laws apply for **all** values of m and n.

ACTIVITY 4.1

Use the law $a^m \times a^n = a^{m+n}$ to answer these.

(i) Write $a^3 \times a^0$ as a single power of a.

Hence state the value of a^0.

(ii) Write $a^2 \times a^{-2}$ as a single power of a.

Hence write a^{-2} in the form $\frac{1}{a^p}$ where p is a positive integer.

(iii) Write $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$ as a single power of a.

Hence copy and complete the statement $a^{\frac{1}{2}}$ is the .... root of a.

(iv) Write $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}}$ as a single power of a.

Hence copy and complete the statement $a^{\frac{1}{3}}$ is the .... root of a.

The following facts should be known.

$$a^0 = 1$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Example 4.24

Express as single powers of x.

(i) $\frac{x^3 \times x^2}{x^9}$ (ii) $\sqrt[3]{x^5 \div x^3}$

(iii) $\sqrt{\frac{x^{\frac{3}{2}} \times x^{\frac{1}{2}}}{(x^3)^2}}$

Solution

(i) $\frac{x^3 \times x^2}{x^9} = \frac{x^{3+2}}{x^9}$

$= \frac{x^5}{x^9}$

$= x^{5-9}$

$= x^{-4}$

(ii) $\sqrt[3]{x^5 \div x^3} = \sqrt[3]{x^{5-3}}$

$= \sqrt[3]{x^2}$

$= (x^2)^{\frac{1}{3}}$

$= x^{2\times\frac{1}{3}}$

$= x^{\frac{2}{3}}$

(iii) $\sqrt{\frac{x^{\frac{3}{2}} \times x^{\frac{1}{2}}}{(x^3)^2}} = \sqrt{\frac{x^{\frac{3}{2}+\frac{1}{2}}}{x^{3\times2}}}$

$= \sqrt{\frac{x^2}{x^6}}$

$= \sqrt{x^{2-6}}$

$= \sqrt{x^{-4}}$

$= (x^{-4})^{\frac{1}{2}}$

$= x^{-4\times\frac{1}{2}}$

$= x^{-2}$

Example 4.25

Solve

(i) $x^{\frac{3}{2}} = 8$

(ii) $x^{-\frac{1}{3}} = 10$

Solution

(i) **Method 1**

Square both sides $\left(x^{\frac{3}{2}}\right)^2 = 8^2$

$x^3 = 64$

Cube root both sides $x = \sqrt[3]{64}$

$x = 4$

Method 2

Take power $\frac{2}{3}$ of both sides $\quad \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = 8^{\frac{2}{3}}$

$$x^1 = (8^2)^{\frac{1}{3}}$$

$$x = 64^{\frac{1}{3}}$$

$$x = \sqrt[3]{64}$$

$$x = 4$$

(ii) **Method 1**

Take the reciprocal of both sides $\quad \left(x^{-\frac{1}{3}}\right)^{-1} = 10^{-1}$

The reciprocal of a^{-m} is a^m → $x^{\frac{1}{3}} = \frac{1}{10}$

Cube both sides $\quad x = \left(\frac{1}{10}\right)^3$

$$x = \frac{1}{1000}$$

Method 2

Cube both sides $\quad \left(x^{-\frac{1}{3}}\right)^3 = 10^3$

$$x^{-1} = 1000$$

Take the reciprocal of both sides $\quad x = 1000^{-1}$ ← The reciprocal of x^{-1} is x

$$x = \frac{1}{1000}$$

Disguised quadratic equations

Some equations, which initially look difficult to solve, can be transformed into quadratic equations which can be solved by factorising or by use of the formula.

An example of such an equation would be $x^4 = 5x^2 - 4$

As it is a quartic equation, the initial reaction is to rearrange it to equal zero, and then search for factors.

However, on closer inspection we notice that there are no odd powers. The substitution $y = x^2$ can be used to transform the equation.

As $x^4 = \left(x^2\right)^2 = y^2$ then the equation can be rewritten as $y^2 = 5y - 4$, which can be solved easily.

$$y^2 - 5y + 4 = 0$$

$$\Rightarrow \quad (y-4)(y-1) = 0$$

$$\Rightarrow \quad y = 4 \text{ or } y = 1$$

but $y = x^2$ $\quad \therefore \quad x^2 = 4 \text{ or } x^2 = 1$

$$\Rightarrow \quad x = \pm 2 \text{ or } x = \pm 1$$

Disguised quadratics are sometimes difficult to spot. However they are easier to spot if the index laws are learnt and practised thoroughly.

Example 4.26

Solve $4^x = 2^x + 56$

> **!** A common error is to treat 4^x as 2×2^x but this is wrong. Instead, 4^x is $2^x \times 2^x$.

Solution

Remembering the index law $(a^m)^n = a^{mn}$, you can rewrite 4^x.

$$4^x = (2^2)^x = 2^{2x} = (2^x)^2$$

So a substitution of $y = 2^x$ should make the question easier to answer.

$$4^x = 2^x + 56$$

$$\Rightarrow \quad (2^x)^2 = 2^x + 56$$

Let $y = 2^x$ $\quad \therefore \quad y^2 = y + 56$

$$\Rightarrow \quad y^2 - y - 56 = 0$$

$$\Rightarrow \quad (y-8)(y+7) = 0$$

$$\Rightarrow \quad y = 8 \quad \text{or} \quad y = -7$$

but $y = 2^x$ $\quad \therefore \quad 2^x = 8 \quad \text{or} \quad 2^x = -7$

but $2^x > 0$ for all values of x

$$\therefore \quad 2^x = 8$$

$$\Rightarrow \quad x = 3 \qquad \text{because } 2^3 = 8$$

Discussion points

- → Which powers, and which roots, can be negative, and which ones can't?
- → Can an exponential function be negative?

> **!** Always check your answers by substituting them back into the original equation. This is good advice when solving any equation, but is particularly good advice in questions such as these.
>
> To avoid rogue answers occurring, remember:
>
> A negative value of x^2 cannot produce a valid solution; but a negative x^3 can.
>
> $\sqrt{x}$ can never be negative; but $\sqrt[3]{x}$ can be negative.
>
> 2^x can never be negative.

Exercise 4F

① Write as single powers of x.

(i) $\dfrac{x \times x^6}{x^3}$

(ii) $x^2 \times x^3 \div x^8$

(iii) $\sqrt{x^7 \div x^2}$

(iv) $x^{\frac{1}{2}} \times x^{\frac{3}{2}} \times x^4$

(v) $\left(x^{\frac{1}{3}}\right)^{12}$

(vi) $\sqrt[3]{x^7 \times x^{-1}}$

(vii) $\sqrt{\dfrac{x^{\frac{5}{2}} \times x^{\frac{1}{2}}}{x^5}}$

(viii) $\left(\sqrt{x}\right)^{10} \div x^{-5}$

(ix) $\sqrt[4]{\dfrac{x^5 \times x^8}{x^3 \times x^2}}$

② Solve, giving solutions as exact values.

(i) $x^{\frac{1}{2}} = 3$

(ii) $x^{\frac{1}{3}} = -2$

(iii) $x^{\frac{1}{2}} = \dfrac{1}{3}$

(iv) $x^{-\frac{1}{2}} = 2$

(v) $x^{-\frac{1}{3}} = 4$

(vi) $x^{-\frac{1}{3}} = 2$

(vii) $x^{\frac{2}{3}} = 4$

(viii) $x^{-\frac{1}{2}} = \dfrac{1}{3}$

(ix) $x^{-2} = \dfrac{9}{25}$

(x) $x^{\frac{3}{2}} = 27$

(xi) $x^{-\frac{1}{3}} = \dfrac{2}{3}$

(xii) $x^{-4} = 10\,000$

③ Expand the following.

(i) $x^{\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)$

(ii) $x^{-2}\left(x^3 - x^2\right)$

(iii) $x^{\frac{2}{3}}\left(x^{\frac{1}{3}} - x^{\frac{7}{3}}\right)$

(iv) $x^{-3}\left(x^{-2} + x^{-1}\right)$

(v) $x^{-\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{7}{2}}\right)$

(vi) $x^{\frac{1}{3}}\left(x^{\frac{8}{3}} - x^{-\frac{1}{3}}\right)$

④ Solve these quadratic equations in $\sqrt{x}$.

(i) $\left(\sqrt{x}\right)^2 - 3\sqrt{x} + 2 = 0$

(ii) $2\left(\sqrt{x}\right)^2 - 5\sqrt{x} + 3 = 0$

(iii) $x - \sqrt{x} - 6 = 0$

(iv) $\sqrt{x} - 6 + \dfrac{5}{\sqrt{x}} = 0$

(v) $2\sqrt{x} + 3 = \dfrac{2}{\sqrt{x}}$

(vi) $\dfrac{3}{\sqrt{x}} + \dfrac{2}{x} = 5$

⑤ Solve these equations in x^2.

(i) $(x^2)^2 - 13x^2 + 36 = 0$ (ii) $x^4 - 15x^2 - 16 = 0$

(iii) $2x^4 - 7x^2 - 4 = 0$ (iv) $4x^4 - x^2 = 18$

(v) $2x^2 = 17 + \frac{9}{x^2}$

PS ⑥ Solve these disguised quadratics.

(i) $x^6 - 7x^3 - 8 = 0$ (ii) $x^{\frac{1}{3}} = 1 + 2x^{-\frac{1}{3}}$

(iii) $(2^x)^2 - 6 \times 2^x + 8 = 0$ (iv) $2^{2x} = 12 \times 2^x - 32$

(v) $9^x = 3^x + 6$ (vi) $4^x = 2^{x+1} + 48$

PS ⑦ Solve the simultaneous equations $5^x + 2^y = 33$ and $5^x - 2^y = 17$.

PS ⑧ Solve these equations.

(i) $2^x = 16$ (ii) $2^{-x} = 8$

(iii) $4^x = 32$ (iv) $9^x = 27$

(v) $27^x = 9^{x+2}$

Hint: Rewrite 27 as 3^3 and 9 as 3^2.

PS ⑨ Solve these index equations.

Hint: Use $x^0 = 1$ and $1^x = 1$.

(i) $(x + 1)^x = 1$ (ii) $(x - 4)^{x+2} = 1$

(iii) $(x - 3)^{x+2} = 1$ (iv) $(x^2 - 5x + 5)^{x+4} = 1$

(v) $(x^2 - 4x + 5)^{x^2-2x-8} = 1$

ACTIVITY 4.2

Create an equation of the form $(ax^2 + bx + c)^{px^2+qx+r} = 1$ which is true for six different values of x.

7 Algebraic proof

Any of the algebraic skills covered in previous sections may be needed in proofs.

When constructing a proof, avoid writing the required result as the first line of working. This is a common error. Instead, start with a given expression and gradually change it using algebraic processes, or start with a known fact which, when combined with other known facts, proves the required result.

Example 4.27

Prove that $2a^3 - a^2(2a - 9)$ is a square number when a is an integer.

Solution

Expand and simplify $\quad 2a^3 - a^2(2a - 9) = 2a^3 - 2a^3 + 9a^2$

$$= 9a^2$$

$$= (3a)^2$$

As a is an integer then $3a$ is also an integer.

$\therefore\ 2a^3 - a^2(2a - 9)$ is a square number when a is an integer.

> The final line of any proof should be a repeat of the required statement in the question.

Example 4.28

(i) Express $x^2 - 8x + 18$ in the form $(x - a)^2 + b$ where a and b are integers.

(ii) Hence, prove that $x^2 - 8x + 18$ is always positive.

Solution

(i) $x^2 - 8x + 18 \equiv (x - a)^2 + b$

$x^2 - 8x + 18 \equiv x^2 - 2ax + a^2 + b$

Equate coefficients of x $\quad -8 = -2a$

$$4 = a$$

Equate constants $\quad 18 = a^2 + b$

$$18 = 16 + b$$

$$2 = b$$

> An alternative is to complete the square of $x^2 - 8x$
> i.e. $x^2 - 8x + 18$
> $= (x - 4)^2 - 4^2 + 18$
> $= (x - 4)^2 + 2$

$x^2 - 8x + 18 \equiv (x - 4)^2 + 2$

(ii) You know that $\quad (x - 4)^2 \geqslant 0$

> When a number is squared the answer can never be negative.

$$\Rightarrow\ (x - 4)^2 + 2 \geqslant 2$$

$$\Rightarrow\ (x - 4)^2 + 2 > 0$$

$$\Rightarrow\ x^2 - 8x + 18 > 0$$

Example 4.29

c and d are positive integers such that $c > d$.

$$f(x) = \frac{2c + cx}{2d + dx} \qquad x \neq -2$$

Prove that $f(x) > 1$

Solution

Factorise the numerator and denominator $\quad f(x) = \dfrac{c(2 + x)}{d(2 + x)}$

Cancel $(2 + x)$ $\quad f(x) = \dfrac{c}{d}$

But $c > d$ (and d is positive) $\Rightarrow\ \dfrac{c}{d} > 1 \quad \therefore\ \ f(x) > 1$

Exercise 4G

1. Prove that $2(m + 7) - 2(5 + m)$ is always a positive integer.
2. Prove that $5(c - 3) + 3(c + 7)$ is always even when c is a positive integer.
3. Prove that $(y + 6)(y + 3) - y^2$ is a multiple of 9 when y is a positive integer.
4. $f(n) = n^2$ for all positive integer values of n.
   - (i) Show that $f(n + 1) = n^2 + 2n + 1$.
   - (ii) Prove that $f(n + 1) + f(n - 1)$ is always even.
   - (iii) Prove that $f(n + 1) - f(n - 1)$ is always a multiple of 4.
5. (i) Express $x^2 + 2x + 5$ in the form $(x + a)^2 + b$ where a and b are integers.
   (ii) Hence, prove that $x^2 + 2x + 5$ is always positive.
6. PS Prove that $y^2 - 10y + 26 > 0$ for all values of y.
7. PS Prove that $9m^2(3m - 1) + (3m)^2$ is a cube number when m is a positive integer.
8. PS Prove that $\dfrac{6p - 18}{2p - 6}$ is always a positive integer when $p \neq 3$
9. PS a is a positive number, b is a negative number.

   $a \neq -b$

   Prove that $\dfrac{a^2 + ab}{ab + b^2}$ is negative.
10. PS $f(x) = x^2 + 2x$.

    Prove that $f(4x) = kx(2x + 1)$ where k is an integer.

FUTURE USES

Students who go on to study mathematics at A-Level will learn a variety of proof techniques, including proof by contradiction and proof by induction.

8 Sequences

Here are the first few terms of a sequence.

4 10 16 22 28 …

The first term is 4. Each subsequent term can be obtained by adding 6 to the previous term.

Here is another sequence

1 4 9 16 25 …

This is the set of square numbers.

The nth term of a sequence is an expression in terms of n.

To find a particular term, a value for n is substituted into the expression.

Example 4.30

Work out the first three terms and the tenth term of the sequence given by

nth term $= n^2 + \dfrac{30}{n}$.

Solution

$$1\text{st term} = 1^2 + \frac{30}{1} = 31$$

$$2\text{nd term} = 2^2 + \frac{30}{2} = 19$$

$$3\text{rd term} = 3^2 + \frac{30}{3} = 19$$

$$10\text{th term} = 10^2 + \frac{30}{10} = 103$$

Example 4.31

The nth term of a sequence is $n^2 - 2n$.

Work out which two consecutive terms have a sum of 179

Solution

$$n\text{th term} = n^2 - 2n$$

$$\Rightarrow \quad (n + 1)\text{th term} = (n + 1)^2 - 2(n + 1)$$

$$\therefore \quad \text{sum of two consecutive terms} = n^2 - 2n + (n + 1)^2 - 2(n + 1) = 179$$

$$\Rightarrow \quad n^2 - 2n + n^2 + 2n + 1 - 2n - 2 = 179$$

$$\Rightarrow \quad 2n^2 - 2n - 180 = 0$$

$$\Rightarrow \quad n^2 - n - 90 = 0$$

$$\Rightarrow \quad (n - 10)(n + 9) = 0$$

$$\Rightarrow \quad n = 10 \quad \text{or} \quad n = -9$$

but n is a number of terms and so must be positive $\therefore\ n = 10$

$\therefore$ the required terms are the 10th term and the 11th term

$$10\text{th term} = 10^2 - 2 \times 10 = 80$$

$$11\text{th term} = 11^2 - 2 \times 11 = 99$$

Linear sequences

The nth term of a linear sequence will be of the form $an + b$ where a and b are constants. Such sequences are sometimes called arithmetic sequences.

Example 4.32

Work out the nth term of the linear sequence

3 12 21 30 39 …

Solution

Method 1

$$n\text{th term} = an + b$$

When $n = 1$, the term is 3 $\quad 3 = a(1) + b$

$$3 = a + b \quad ①$$

When $n = 2$, the term is 12 $\quad 12 = a(2) + b$

$12 = 2a + b$ ②

Subtracting ① from ② $\quad 9 = a$

Substituting in ① $\quad 3 = 9 + b$

$-6 = b$

nth term $= 9n - 6$

Method 2

nth term $= an + b$

The common difference between the terms is $\quad 12 - 3 = 9$

This will be the coefficient of n $\quad a = 9$

When $n = 1$ the term is 3 $\quad 3 = 9(1) + b$

$3 = 9 + b$

$-6 = b$

nth term $= 9n - 6$

Example 4.33

Work out the nth term of the linear sequence

15 13 11 9 …

Note

For a linear sequence which decreases, the common difference is negative.

Solution

The common difference is $13 - 15 = -2$

Find a number C such that $-2 \times 1 + C = 15$

C must be 17

So the nth term is $-2n + 17$

(which can also be written as $17 - 2n$)

Exercise 4H

① Work out an expression for the nth term for each of the following linear sequences.

(i) 10 14 18 22 26 …
(ii) 2 9 16 23 …
(iii) −5 −3 −1 1 3 …
(iv) 0 25 50 75 …
(v) −11 −3 5 13 …
(vi) 3 3.5 4 4.5 5 …
(vii) 40 30 20 10 0 −10 …
(viii) 7 4 1 −2 −5 …
(ix) $1 \quad \frac{1}{2} \quad 0 \quad -\frac{1}{2} \quad -1 \quad \ldots$
(x) −4 −5.5 −7 −8.5 −10 …

② (i) Work out the 100th term of the linear sequence −5 1 7 13 …
(ii) Work out the 50th term of the linear sequence 35 28 21 14 …
(iii) Work out the 200th term of the linear sequence −1 −10 −19 −28 …

PS ③ Here is a linear sequence.

3   5.5   8   10.5   13   …

Work out the value of the first term of the sequence that is greater than 250.

PS ④ Here are two linear sequences.

Sequence A   7   5   3   1   −1   …

Sequence B   3   0   −3   −6   …

Prove that the sum of the nth terms of the two sequences is $15 - 5n$.

PS ⑤ Here is a linear sequence.

$p \quad p + 2q \quad p + 4q \quad p + 6q \quad \ldots$

The third term of the sequence is 20

The fourth term of the sequence is 56

(i) Work out the values of p and q.

(ii) Work out an expression for the nth term of the sequence.

PS ⑥ The nth term of a sequence is $7n - 3$

Explain why 44 is not a term in the sequence.

PS ⑦ The nth term of a sequence is n^3.

Prove that the difference between consecutive terms is never a multiple of 3

PS ⑧ The nth term of a sequence is $n^2 - 40n + 405$.

Prove that every term is positive.

Quadratic sequences

The nth term of a quadratic sequence is of the form $an^2 + bn + c$ where a, b and c are constants.

One method for finding the nth term of a quadratic sequence is to work out the difference between the differences.

Consider three consecutive terms:

$$(n-1)\text{th term} = a(n-1)^2 + b(n-1) + c = an^2 + bn + c - 2an + a - b$$

$$n\text{th term} = an^2 + bn + c$$

$$(n+1)\text{th term} = a(n+1)^2 + b(n+1) + c = an^2 + bn + c + 2an + a + b.$$

The difference between the nth term and the $(n-1)$th term is $2an - a + b$.

The difference between the $(n+1)$th term and the nth term is $2an + a + b$.

The difference between consecutive differences is then $2a$.

So the coefficient of n^2 (a) is always half of the second difference.

Example 4.34

Work out the nth term of the quadratic sequence   3   6   13   24   39   …

Solution

nth term $= an^2 + bn + c$

Work out the second differences.

For a quadratic sequence, all the second differences will be the same

3		6		13		24		39
	3		7		11		15	
		4		4		4		

The coefficient a is half of the second difference.

In this case a = half of 4 = 2.

Then compare the sequence $2n^2$ with the original sequence.

Original sequence =	3	6	13	24	39
$2n^2$ =	2	8	18	32	50

What needs to be added to each of the $2n^2$ sequence to make the original sequence?

1   −2   −5   −8   −11

Notice that this sequence is linear and its nth term is $-3n + 4$

The required quadratic sequence is the sum of the two sequences $2n^2$ and $-3n + 4$

nth term $= 2n^2 - 3n + 4$

Another method for finding the nth term of a quadratic sequence is shown in the final section of this chapter, in Example 4.38.

Exercise 4I

1. Work out the nth term for each of the following quadratic sequences.
   (i) 4 9 16 25 36 …
   (ii) 0 6 14 24 36 …
   (iii) 4 13 24 37 52 …
   (iv) 8 21 40 65 96 …
   (v) 4 13 26 43 64 …
   (vi) −4 −4 0 8 20 …
   (vii) 11 10 7 2 −5 …
   (viii) 98 92 82 68 50 …
2. (i) Work out the nth term of the linear sequence
   1 5 9 13 17 …
   (ii) Hence work out the nth term of the quadratic sequence
   1 25 81 169 289 …
   Give your answer in the form $an^2 + bn + c$.
3. (i) Work out the nth term of the quadratic sequence
   2 7 14 23 34 …
   (ii) Hence work out the nth term of the quadratic sequence
   5 10 17 26 37 …
   Give your answer in the form $an^2 + bn + c$.
4. (i) Work out the nth term of the quadratic sequence
   −5 −6 −5 −2 3 …

(ii) Hence work out the nth term of the quadratic sequence

$-15 \quad -18 \quad -15 \quad -6 \quad 9 \quad \ldots$

Give your answer in the form $an^2 + bn + c$.

(iii) Hence work out the nth term of the quadratic sequence

$0 \quad -3 \quad 0 \quad 9 \quad 24 \quad \ldots$

Give your answer in the form $an^2 + bn + c$.

⑤ A sequence starts with 2 and then follows the rule 'double the previous term and add 3' to generate subsequent terms. The first three terms are 2, 7, 17

(i) Calculate the 5th term.

(ii) How many terms will be even? Explain your answer.

⑥ The 1st, 3rd and 5th terms of a quadratic sequence are 11, 15 and 27 respectively.

Work out the nth term of the sequence.

⑦ The 2nd, 3rd and 4th terms of a quadratic sequence are −4, −1, and 4 respectively.

Work out the nth term of the sequence.

ACTIVITY 4.3

Draw n points on the edge of a circle – not equally spaced.
Join every pair of points with a straight line in such a way that no more than two lines intersect at any position inside the circle. You may have to move one (or more) points to avoid this happening.
Count the number of regions inside the circle for each value of n.

For $n = 1$ the number of regions is 1.
For $n = 2$ the number of regions is 2.
For $n = 3$ the number of regions is 4.
For $n = 4$ the number of regions is 8.
For $n = 5$ the number of regions is 16.

The sequence seems obvious!
Make a prediction for $n = 6$ and check it!

As the mathematician Eric Bell commented during the early twentieth century, *'Obvious is the most dangerous word in mathematics'*.

9 Limiting value of a sequence

ACTIVITY 4.4

A sequence is given by nth term $= \dfrac{3n}{n+1}$.

(i) Work out the first 15 terms of the sequence (giving your answers to 3 decimal places where necessary).

(ii) Work out the 20th, 30th, 40th, 50th, 100th, 200th and 500th terms of the sequence.

(iii) Explain what is happening to the terms as n increases.

Limiting value of a sequence

To find the limiting value of a sequence, consider the nth term as $n \to \infty$ (n becomes very large).

Example 4.35

The nth term of a sequence is $\frac{2n-1}{3n+2}$.

Prove that the limiting value of the sequence as $n \to \infty$ is $\frac{2}{3}$.

Solution

Divide numerator and denominator by n.

$$\frac{2n-1}{3n+2} = \frac{\frac{2n}{n} - \frac{1}{n}}{\frac{3n}{n} + \frac{2}{n}} = \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}}$$

but as n gets bigger (approaching infinity) then $\frac{1}{n}$ and $\frac{2}{n}$ both get smaller, (they both approach zero).

∞ is the symbol for infinity.

This is written like this: as $n \to \infty$ then $\frac{1}{n} \to 0$ and $\frac{2}{n} \to 0$

So as $n \to \infty$ then $\frac{2 - \frac{1}{n}}{3 + \frac{2}{n}} \to \frac{2-0}{3+0} = \frac{2}{3}$

So the limiting value of $\frac{2n-1}{3n+2}$ is $\frac{2}{3}$

! Do not write $\frac{1}{n} = 0$ as this is wrong. Zero is the **limit** as n approaches infinity but, as n can never get to infinity, then $\frac{1}{n}$ can never get to zero.

Likewise, do not write nth term $= \frac{2}{3}$, as it doesn't. $\frac{2}{3}$ is the limit.

Exercise 4J

① The nth term of a sequence is $\frac{n+1}{2n+1}$.

(i) Work out the first three terms of the sequence.

(ii) Work out the position of the term that has value 0.52.

② The nth term of a sequence is $\frac{4n-1}{2n-5}$.

(i) Work out the position of the term that has value 2.36.

(ii) Show that 1 is not a term in the sequence.

③ The nth terms of sequences are shown.

Work out the limiting value of each sequence as $n \to \infty$.

(i) $\frac{2n}{n+1}$ (ii) $\frac{n+2}{n+3}$ (iii) $\frac{n}{3n-1}$

(iv) $\frac{2n-1}{4n+1}$ (v) $\frac{3n}{4n+1}$ (vi) $\frac{1-n}{n+4}$

(vii) $\frac{2n}{3-4n}$ (viii) $\frac{2-6n}{5-2n}$

④ The nth term of a sequence is $\frac{5n+1}{2n+1}$.

Prove that the limiting value of the sequence as $n \to \infty$ is $\frac{5}{2}$.

⑤ The nth term of a sequence is $\frac{10-6n}{8n-3}$.

Prove that the limiting value of the sequence as $n \to \infty$ is -0.75

PS ⑥ The nth term of a sequence is $2 + \frac{7}{an+b}$, $a \neq 0$

Work out its limiting value.

PS ⑦ The nth term of a sequence is $\frac{an+3}{cn-1}$.

The 1st term of the sequence is 11, and its limiting value as $n \to \infty$ is 4

Work out the value of the 2nd term of the sequence.

PS ⑧ The nth term of a sequence is $\frac{2n^2+3n-4}{7n^2-n+2}$.

Work out its limiting value.

(Hint: divide the numerator and denominator by n^2.)

Prior knowledge

Students should already have a good understanding of simultaneous equations in two unknowns.

10 Simultaneous equations in three unknowns

Three equations are required when solving simultaneous equations in three unknowns.

Step 1: Eliminate one unknown by combining a pair of equations.

Step 2: Eliminate the same unknown by combining another pair of equations.

Step 3: Solve the resulting pair of equations using the method described in section 4.2.

Step 4: Substitute these two unknowns into one of the original equations to work out the value of the third unknown.

Example 4.36

Solve the three equations

$3x + 2y - 3z = -13$, $2x - 3y + 4z = 24$ and $4x - 5y + 2z = 22$

Solution

$3x + 2y - 3z = -13$ ①

$2x - 3y + 4z = 24$ ②

$4x - 5y + 2z = 22$ ③

Step 1:

$12x + 8y - 12z = -52$ — $4 \times$ equation ①

$6x - 9y + 12z = 72$ — $3 \times$ equation ②

Adding: $18x - y = 20$ ④

Step 2:

$8x - 10y + 4z = 44$ — $2 \times$ equation ③

$2x - 3y + 4z = 24$ — equation ②

Subtracting: $6x - 7y = 20$ ⑤

Step 3: $18x - y = 20$ equation ④

$18x - 21y = 60$ $3 \times$ equation ⑤

Subtracting: $20y = -40$

$y = -2$

Substitute $y = -2$ into equation ④: $18x - (-2) = 20$

$18x = 18$

$x = 1$

Step 4: Substitute $x = 1$ and $y = -2$ into equation ①:

$3 \times 1 + 2 \times -2 - 3z = -13$

$-3z = -12$

$z = 4$

$\therefore \quad x = 1, \quad y = -2 \quad \text{and} \quad z = 4$

Example 4.37

Solve the three equations

$z - 8 = 2x + 3y$, $3x - 21 = 2y - 2z$ and $2x + y = 3z - 4$

Solution

$z - 8 = 2x + 3y$ ①

$3x - 21 = 2y - 2z$ ②

$2x + y = 3z - 4$ ③

Step 1: Rearrange equation ①: $z = 2x + 3y + 8$

Substitute for z in equation ②: $3x - 21 = 2y - 2(2x + 3y + 8)$

$3x - 21 = 2y - 4x - 6y - 16$

$7x + 4y = 5$ ④

Step 2: Substitute for z in equation ③: $2x + y = 3(2x + 3y + 8) - 4$

$2x + y = 6x + 9y + 24 - 4$

$-4x - 8y = 20$

$x + 2y = -5$ ⑤

Step 3: Rearrange equation ⑤: $x = -2y - 5$

Substitute for x in equation ④: $7(-2y - 5) + 4y = 5$

$-14y - 35 + 4y = 5$

$-10y = 40$

$y = -4$

Substitute $y = -4$ into equation ⑤: $x + 2 \times -4 = -5$

$x - 8 = -5$

$x = 3$

Step 4: Substitute $x = 3$ and $y = -4$ into $z = 2x + 3y + 8$

$$z = 2 \times 3 + 3 \times -4 + 8$$
$$z = 6 - 12 + 8$$
$$z = 2$$

$\therefore \quad x = 3, \quad y = -4 \quad \text{and} \quad z = 2$

Example 4.38

0, 3, 10 are the first three terms of a sequence with nth term $= an^2 + bn + c$.

(i) By substituting $n = 1, 2$ and 3, set up three simultaneous equations in a, b and c.

(ii) Solve your simultaneous equations to find an expression for the nth term.

Solution

(i) $a + b + c = 0$ ①

$4a + 2b + c = 3$ ②

$9a + 3b + c = 10$ ③

(ii) ② − ① $3a + b = 3$ ④

③ − ② $5a + b = 7$ ⑤

⑤ − ④ $2a = 4$

$a = 2$

Substitute $a = 2$ into ④ $6 + b = 3$

$b = -3$

Substitute $a = 2$ and $b = -3$ into ① $2 + (-3) + c = 0$

$c = 1$

The nth term is $2n^2 - 3n + 1$

Exercise 4K

① (i) Solve, by eliminating z:

$$2x + 3y + z = 12$$
$$3x + 2y + z = 13$$
$$4x - 5y + z = 8$$

(ii) Solve, by eliminating y:

$$3x + y - 2z = 4$$
$$5x - y + 3z = 22$$
$$2x + y + 4z = 13$$

(iii) Solve, by eliminating x:

$$2x - 3y + 4z = -7$$
$$2x + 2y - 3z = 19$$
$$2x - 5y + 2z = -3$$

② (i) Solve $x + 2y + 3z = 13$
$2x + 3y - z = -7$
$3x - y + 2z = 18$

(ii) Solve $2x - y + 2z = 16$
$3x + 2y - z = 5$
$x + 4y - 3z = -13$

(iii) Solve $4x + 2y - z = 29$
$-x + 3y + 2z = -16$
$2x + y - 3z = 22$

(iv) Solve $5x + 3y + 2z = 8$
$7x - 4y + 4z = 20$
$3x + 2y - 2z = 3$

③ (i) Solve $2a - 5b + 2c = -36$, $3a + 4b - 3c = 10$ and $4a - 3b + 4c = -44$

(ii) Solve $3p + 5q - 2r = 13$, $4p - 2q + 5r = -25$ and $-2p + 3q - 7r = 25$

(iii) Solve $2\alpha + 3\beta + 2\gamma = 1$, $4\alpha - 2\beta - 5\gamma = 15$ and $-5\alpha + 4\beta + \gamma = -42$

④ (i) Solve $x + 3y = 2z - 23$, $2y = 3x - 4z + 2$ and $5z - 2x = 3y + 37$

(ii) Solve $2x = y + z + 13$, $5y - x = 2z - 16$ and $3z + 2y = 11 - x$.

(iii) Solve $y + 5 = 3x$, $2x - z = 7$ and $4y = 3z + 13$

PS ⑤ $x = 3$, $y = 5$, $z = 1$ is the solution to the simultaneous equations
$ax + by + cz = 28$
$az - cy = 2bx + 10$
$bz = ax + 3cy + 26$

Work out the values of a, b and c.

PS ⑥ 7, 9, 13 are the first three terms of a sequence with nth term $= an^2 + bn + c$.

(i) By substituting $n = 1, 2$ and 3 set up three simultaneous equations in a, b and c.

(ii) Solve your simultaneous equations and hence write down an expression for the nth term.

PS ⑦ 9, 16 and 42 are the 2nd, 3rd and 5th terms of a quadratic sequence. Work out the nth term of the sequence.

PS ⑧ If (x, y, z) are the general coordinates of a point in a 3-dimensional coordinate system, then each of the equations

$5x - 2y + z = 3$
$z = x + y$
$2x + 6y = 5z$

represents a plane. Solve the equations and hence write down the coordinates of the point at which the three planes meet.

Students of Level 2 Further Mathematics will not need to have any understanding of plane geometry. This question is for interest only.

ACTIVITY 4.5

The following questions are beyond the specification for Level 2 Further Mathematics.
However, students could use a 3-D graph plotter, along with three pieces of card, to discover the answers.

(i) Under what circumstances would two planes never meet?

(ii) If two planes meet, how many points lie on both planes?

The general equation for a plane is $ax + by + cz = d$ where a, b, c and d are constants.

(iii) Write an equation for a plane which is parallel to the plane $3x + 2y - z = 5$

(Hint: Use a 3-D graph plotter and see what happens as each coefficient is changed.)

(iv) Do three non-parallel planes always share a common point?

(v) Is it possible to have more than one common point on three different planes?

FUTURE USES

Students who go on to study Further Mathematics at A-Level will learn more about plane geometry.

REAL-WORLD CONTEXT

Simultaneous equations and inequalities involving multiple variables are used when solving real-life problems. Given a variety of constraints, cost can be minimised and profit maximised, along with other optimisation situations. Such problems can be solved using linear programming techniques which are studied in A-Level Further Mathematics.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- solve quadratic equations
  - by factorising
  - by completing the square
  - using the quadratic formula
  - by drawing a graph
- solve simultaneous equations
  - in two unknowns
  - in three unknowns
  - by plotting their graphs
- use the factor theorem
  - to factorise a polynomial
  - to solve a polynomial equation
- solve inequalities
  - linear
  - quadratic
- use the index laws
- prove mathematical statements algebraically
- find and use the nth term of sequences
  - linear
  - quadratic
- find the limit of a sequence with an nth term of the form $\frac{an + b}{cn + d}$.

KEY POINTS

1 When factorising quadratics of the form $ax^2 + bx + c$, find two numbers with a sum of b and product ac. Then split the coefficient of x into these two numbers.

2 When completing the square on a quadratic of the form $ax^2 + bx + c$, take a out as a factor, or divide both sides of the equation by a.

3 If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

4 $f(a) = 0 \quad \Leftrightarrow \quad (x - a)$ is a factor of $f(x)$

5 When solving an inequality, treat it as an equation, but remember to reverse the sign if the inequality is multiplied (or divided) throughout by a negative value.

6 When solving a quadratic inequality, first take all terms to one side.

7 The index laws are:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$

8 Some useful results are:

- $a^0 = 1$
- $a^{-m} = \frac{1}{a^m}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

9 In a proof question, show every step clearly.

10 A linear sequence has an nth term of the form $an + b$.

11 A quadratic sequence has an nth term of the form $an^2 + bn + c$.

12 A sequence with an nth term of the form $\frac{an + b}{cn + d}$ has a limit of $\frac{a}{c}$ as n approaches infinity.

5 Coordinate geometry

Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone.

Albert Einstein

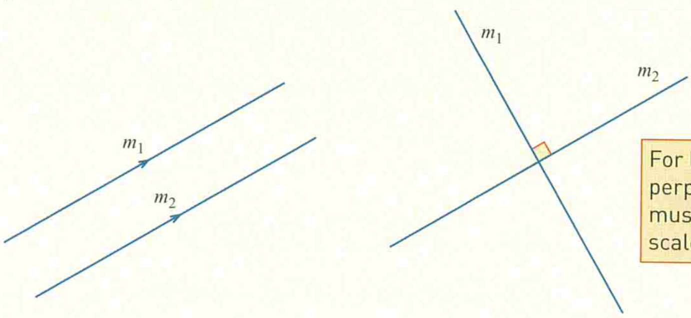
1 Parallel and perpendicular lines

Prior knowledge

In Chapter 3 we used this fact:

The line joining (x_1, y_1) to (x_2, y_2) has gradient m, where $m = \frac{y_2 - y_1}{x_2 - x_1}$.

If you know the gradients m_1 and m_2 of two lines, you can tell at once if they are parallel or perpendicular.



For lines to look perpendicular you must use the same scale on both axes.

Figure 5.1

Parallel lines: $m_1 = m_2$ Perpendicular lines: $m_1 m_2 = -1$

To illustrate the result for perpendicular lines, try Activity 5.1 on squared paper.

Discussion point

➜ How would you explain the result for parallel lines?

ACTIVITY 5.1

(i) Draw two congruent right-angled triangles in the positions shown in Figure 5.2. p and q can take any value.

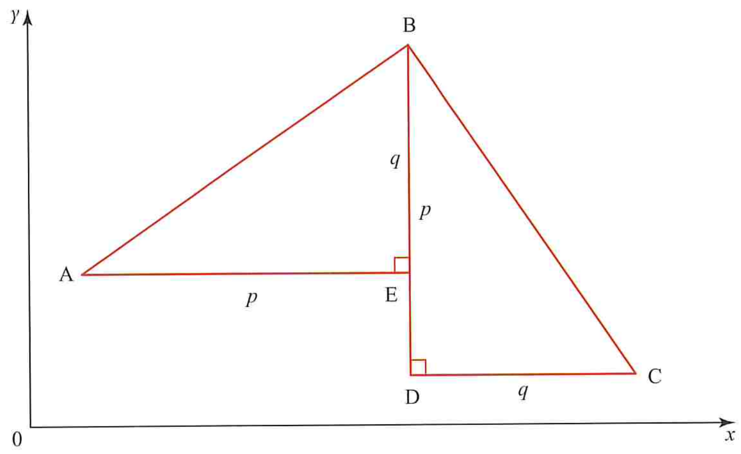


Figure 5.2

(ii) Explain why $\angle ABC = 90°$.

(iii) Calculate the gradient of AB (m_1) and the gradient of BC (m_2).

(iv) Show that $m_1 m_2 = -1$

! The gradient of a line perpendicular to a line of gradient m is given by $-\frac{1}{m}$.
Don't forget to change the sign when taking the reciprocal.

2 The distance between two points

Prior knowledge

Students are expected to use Pythagoras' theorem to calculate the distance between two points with known coordinates.

Look at Figure 5.3. P is (3, 1) and Q is (6, 5).

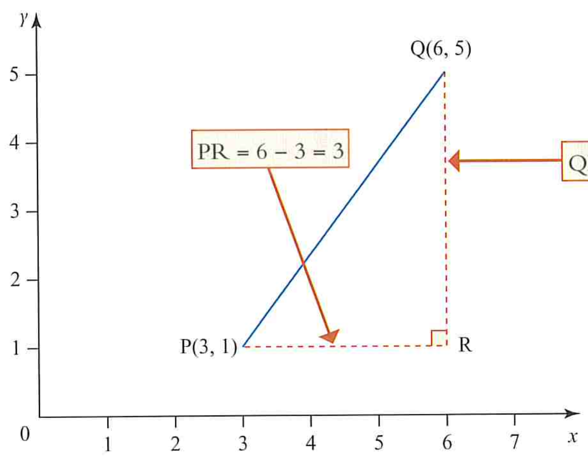


Figure 5.3

$$PQ = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Generalising this, if P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then

length $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

3 The midpoint of a line joining two points

Look at the line joining the points P(1, 2) and Q(7, 4) in Figure 5.4. The point M is the midpoint of PQ.

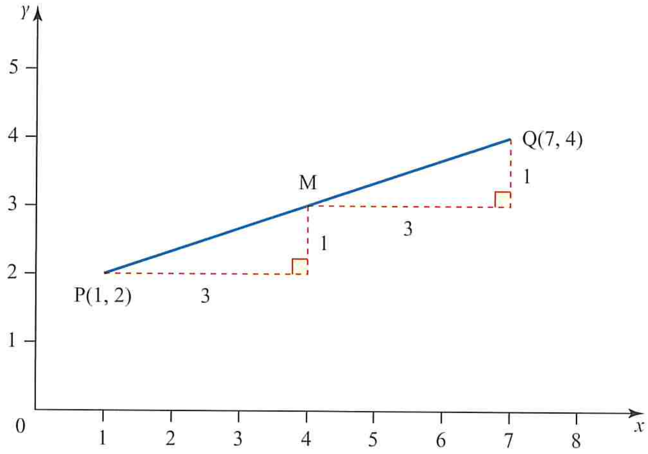


Figure 5.4

The coordinates of M are the means (averages) of the coordinates of P and Q.

$$\frac{1}{2}(1 + 7) = 4 \text{ and } \frac{1}{2}(2 + 4) = 3$$

M is $(4, 3)$.

Again, if P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then the coordinates of the midpoint of PQ are given by

$$\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 5.1

A and B are the points (−4, 2) and (2, 5). Work out

(i) the gradient of AB

(ii) the gradient of the line perpendicular to AB

(iii) the length of AB

(iv) the coordinates of the midpoint of AB.

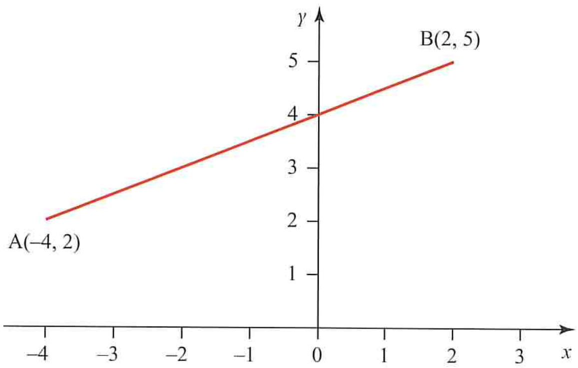


Figure 5.5

Solution

(i) Taking (−4, 2) as (x_1, y_1) and (2, 5) as (x_2, y_2)

$$\text{gradient} = \frac{5-2}{2-(-4)} = \frac{3}{6} = \frac{1}{2}$$

(ii) $m_1 = \frac{1}{2}$ and $m_1 m_2 = -1$

$$\Rightarrow \quad \frac{1}{2}m_2 = -1$$

$$\Rightarrow \quad m_2 = -2$$

The line perpendicular to AB has gradient −2

(iii)
$$\begin{aligned} \text{length} &= \sqrt{(2-(-4))^2 + (5-2)^2} \\ &= \sqrt{36+9} \\ &= \sqrt{45} \\ &= 6.71 \text{ (3 s.f.)} \end{aligned}$$

(iv)
$$\begin{aligned} \text{midpoint} &= \left(\frac{-4+2}{2}, \frac{2+5}{2}\right) \\ &= (-1, 3.5) \end{aligned}$$

Example 5.2

P is the point (a, b) and Q is the point $(3a, 5b)$.

Write expressions, in terms of a and b, for

(i) the gradient of PQ

(ii) the length of PQ

(iii) the midpoint of PQ.

Solution

Taking (a, b) as (x_1, y_1) and $(3a, 5b)$ as (x_2, y_2)

(i)
$$\begin{aligned} \text{gradient} &= \frac{5b-b}{3a-a} \\ &= \frac{4b}{2a} = \frac{2b}{a} \end{aligned}$$

(ii)
$$\begin{aligned} \text{length} &= \sqrt{(3a-a)^2 + (5b-b)^2} \\ &= \sqrt{4a^2 + 16b^2} \end{aligned}$$

(iii)
$$\begin{aligned} \text{midpoint} &= \left(\frac{a+3a}{2}, \frac{b+5b}{2}\right) \\ &= (2a, 3b) \end{aligned}$$

Discussion point

➜ How can the length in part (ii) be simplified further?

Example 5.3

A, B and C are the points (1, 2), (5, *b*) and (6, 2). $\angle ABC = 90°$.

(i) Work out two possible values of *b*.

(ii) Show all four points on a sketch and describe the shape of the figure you have drawn.

Solution

(i) Gradient of AB $= \dfrac{b-2}{5-1} = \dfrac{b-2}{4}$

Gradient of BC $= \dfrac{2-b}{6-5} = 2-b$

$$\angle ABC = 90° \Rightarrow \text{AB and BC are perpendicular}$$

$$\Rightarrow \left(\frac{b-2}{4}\right) \times (2-b) = -1$$

$$\Rightarrow (b-2)(2-b) = -4$$

$$\Rightarrow 2b - b^2 - 4 + 2b = -4$$

$$\Rightarrow 4b - b^2 = 0$$

$$\Rightarrow b(4-b) = 0$$

So $b = 0$ or $b = 4$.

(ii)

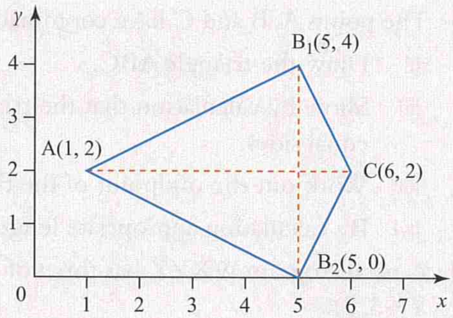


Figure 5.6

AB_1CB_2 is a quadrilateral with diagonals that are perpendicular, since AC is parallel to the *x*-axis and B_1B_2 is parallel to the *y*-axis.

This makes AB_1CB_2 a kite.

Exercise 5A

① For each of the following pairs of points A and B, calculate

(a) the gradient of the line perpendicular to AB

(b) the length of AB

(c) the coordinates of the midpoint of AB.

(i) A(4, 3) B(8, 11)

(ii) A(3, 4) B(0, 13)

(iii) A(5, 3) B(10, −8)

(iv) A$(-6, -14)$ B$(1, 7)$
(v) A$(6, 0)$ B$(8, 15)$
(vi) A$(-2, -4)$ B$(3, 9)$
(vii) A$(-3, -6)$ B$(2, -7)$
(viii) A$(4, 7)$ B$(7, -4)$

② A$(0, 5)$, B$(4, 1)$ and C$(2, 7)$ are the vertices of a triangle. Show that the triangle is right-angled
(i) by finding the gradients of the sides
(ii) by finding the lengths of the sides.

③ A$(3, 6)$, B$(7, 4)$ and C$(1, 2)$ are the vertices of a triangle. Show that ABC is a right-angled isosceles triangle.

④ A$(3, 5)$, B$(3, 11)$ and C$(6, 2)$ are vertices of a triangle.
(i) Work out the perimeter of the triangle.
(ii) Using AB as the base, work out the area of the triangle.

⑤ A quadrilateral PQRS has vertices at P$(-2, -5)$, Q$(11, -7)$, R$(9, 6)$ and S$(-4, 8)$.
(i) Work out the lengths of the four sides of PQRS.
(ii) Work out the midpoints of the diagonals PR and QS.
(iii) Without drawing a diagram, show why PQRS cannot be a square. What is it?

⑥ The points A, B and C have coordinates $(2, 3)$, $(6, 12)$ and $(11, 7)$ respectively.
(i) Draw the triangle ABC.
(ii) Show by calculation that the triangle is isosceles and name the two equal sides.
(iii) Work out the midpoint of the third side.
(iv) By calculating appropriate lengths, work out the area of triangle ABC.

⑦ A parallelogram WXYZ has three of its vertices at W$(2, 1)$, X$(-1, 5)$ and Y$(-3, 3)$.
(i) Work out the midpoint of WY.
(ii) Use this information to work out the coordinates of Z.

⑧ A triangle ABC has vertices at A$(3, 2)$, B$(4, 0)$ and C$(8, 2)$.
(i) Show that the triangle is right-angled.
(ii) Work out the coordinates of the point D such that ABCD is a rectangle.

⑨ The three points P$(-2, 3)$, Q$(1, q)$ and R$(7, 0)$ are collinear (i.e. they lie on the same straight line).
(i) Work out the value of q.
(ii) Work out the ratio of the lengths PQ : QR.

⑩ A quadrilateral has vertices A$(-2, 8)$, B$(-5, 5)$, C$(5, 3)$ and D$(3, 7)$.
(i) Draw the quadrilateral.
(ii) Show by calculation that it is a trapezium.
(iii) Work out the coordinates of E when ABCE is a parallelogram.

4 Equation of a straight line

Prior knowledge

In Chapter 3 you used these three facts.

- The equation of the line with gradient m cutting the y-axis at the point $(0, c)$ is $y = mx + c$.
- The equation of the line with gradient m passing through (x_1, y_1) is $y - y_1 = m(x - x_1)$.
- The equation of the line passing through (x_1, y_1) and (x_2, y_2) is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Example 5.4

An isosceles triangle with AB = AC has vertices at A(2, 3), B(8, 5) and C(4, 9).

Work out the equation of the line of symmetry.

Solution

Figure 5.7 shows the triangle ABC with the line of symmetry joining A to the midpoint of BC.

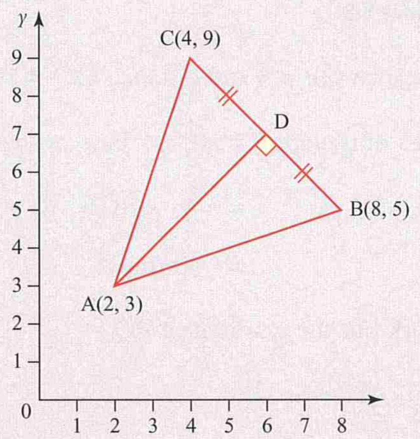


Figure 5.7

The coordinates of D are $\left(\dfrac{8+4}{2}, \dfrac{5+9}{2}\right) = (6, 7)$

Let (x_1, y_1) be (2, 3) and (x_2, y_2) be (6, 7).

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \quad \frac{y - 3}{7 - 3} = \frac{x - 2}{6 - 2}$$

$$\Rightarrow \quad \frac{y - 3}{4} = \frac{x - 2}{4}$$

$$\Rightarrow \quad y = x + 1$$

Example 5.5

The straight line with equation $5x - 4y = 40$ intersects the x-axis at P and the y-axis at Q.

(i) Work out the area of triangle OPQ where O is the origin.

(ii) Work out the equation of the line that passes through Q and is perpendicular to PQ.

Solution

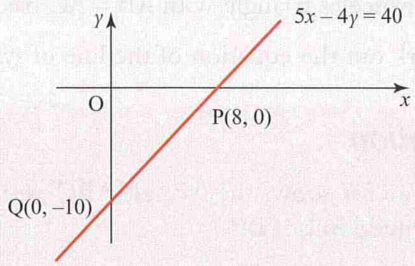
(i) Work out the coordinates of P and Q.

Substitute $y = 0$ in equation of line $\quad 5x - 0 = 40$

$x = 8 \qquad$ P(8, 0)

Substitute $x = 0$ in equation of line $\quad 0 - 4y = 40$

$y = -10 \qquad$ Q(0, −10)



A sketch graph will often be useful.

Figure 5.8

Distance OP = 8 and distance OQ = 10.

$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times 8 \times 10$$
$$= 40 \text{ units}^2$$

(ii) Work out the gradient of PQ

$$\frac{0 - (-10)}{8 - 0} = \frac{10}{8}$$
$$= \frac{5}{4}$$

Gradient of line perpendicular to PQ $= -\frac{4}{5}$

Line passes through (0, −10) $\quad y = -\frac{4}{5}x - 10$

Exercise 5B

① By calculating the gradients of the following pairs of lines, state whether they are parallel, perpendicular or neither.

(i) $x = 2$
$y = -2$

(ii) $y = 2x$
$y = -2x$

(iii) $x + 2y = 1$
$2x - y = 1$

(iv) $y = x - 3$
$x - y + 4 = 0$

(v) $y = 3 - 4x$
$y = 4 - 3x$

(vi) $x + y = 5$
$x - y = 5$

(vii) $x - 2y = 3$
$y = \frac{1}{2}x - 1$

(viii) $x + 3y - 4 = 0$
$y = 3x + 4$

(ix) $2y = x$
$2x + y = 4$

(x) $2x + 3y - 4 = 0$
$2x + 3y - 6 = 0$

(xi) $x + 3y = 1$
$y + 3x = 1$

(xii) $2x = 5y$
$5x + 2y = 0$

② Work out the equations of these lines.

(i) Parallel to $y = 3x$ and passing through $(3, -1)$.
(ii) Parallel to $y = 2x + 3$ and passing through $(0, 7)$.
(iii) Parallel to $y = 3x - 4$ and passing through $(3, -7)$.
(iv) Parallel to $4x - y + 2 = 0$ and passing through $(5, 0)$.
(v) Parallel to $3x + 2y - 1 = 0$ and passing through $(3, -2)$.
(vi) Parallel to $2x + 4y - 5 = 0$ and passing through $(0, 5)$.

③ Work out the equations of these lines.

(i) Perpendicular to $y = 2x$ and passing through $(0, 0)$.
(ii) Perpendicular to $y = 3x - 1$ and passing through $(0, 4)$.
(iii) Perpendicular to $y + x = 2$ and passing through $(3, -1)$.
(iv) Perpendicular to $2x - y + 4 = 0$ and passing through $(1, -1)$.
(v) Perpendicular to $3x + 2y + 4 = 0$ and passing through $(3, 0)$.
(vi) Perpendicular to $2x + y - 1 = 0$ and passing through $(4, 1)$.

④ Points P and Q have coordinates P$(3, -1)$ and Q$(5, 7)$.

(i) Work out the gradient of PQ.
(ii) Work out the coordinates of the midpoint of PQ.
(iii) The perpendicular bisector of a line PQ is the line which is perpendicular to PQ and passes through its midpoint. Work out the equation of the perpendicular bisector of PQ.

⑤ A triangle has vertices P$(2, 5)$, Q$(-2, -2)$ and R$(6, 0)$.

(i) Sketch the triangle.
(ii) Work out the coordinates of L, M and N, which are the midpoints of PQ, QR and RP respectively.
(iii) Work out the equations of the lines LR, MP and NQ (these are the medians of the triangle).
(iv) Show that the point $(2, 1)$ lies on all three of these lines. (This shows that the medians of a triangle are concurrent.)

⑥ The straight line with equation $2x + 3y - 12 = 0$ cuts the x-axis at A and the y-axis at B.

(i) Sketch the line.

(ii) Work out the coordinates of A and B.

(iii) Work out the area of triangle OAB where O is the origin.

(iv) Work out the equation of the line which passes through O and is perpendicular to AB.

(v) Work out the length of AB and, using the result in (iii), calculate the shortest distance from O to AB.

⑦ A quadrilateral has vertices at the points A(−7, 0), B(2, 3), C(5, 0) and D(−1, −6).

(i) Sketch the quadrilateral.

(ii) Work out the gradient of each side.

(iii) Work out the equation of each side.

(iv) Work out the length of each side.

(v) Work out the area of the quadrilateral.

⑧ £10 000 is invested and simple interest of 2% per annum is received on this investment. (Simple interest is when the interest received each year is calculated on the initial investment in the account only.)

(i) Calculate the interest received after each of the first three years.

(ii) Sketch the graph of interest against time and write down its equation.

(iii) Use the equation to work out how long it would take for the investment to reach £11 000

⑨ A spring has an unstretched length (often called the natural length) of 20 cm. When it is hung with a load of 50 g attached, its stretched length is 25 cm.

Assuming that the extension of the spring is proportional to the load at all times

(i) calculate the load corresponding to an extension of 12.5 cm

(ii) calculate the extension corresponding to a load of 75 g

(iii) calculate the extension corresponding to a load of 800 g and comment on your answer.

Prior knowledge

Students learned to solve simultaneous linear equations in Chapter 4.

5 The intersection of two lines

You can work out the point of intersection of any two lines (or curves) by solving their equations simultaneously.

Example 5.6

(i) Sketch the lines $x + 3y - 6 = 0$ and $y = 2x - 5$ on the same axes.

(ii) Work out the coordinates of the point where they intersect.

Solution

(i) The line $x + 3y - 6 = 0$ passes through (0, 2) and (6, 0).

The line $y = 2x - 5$ passes through (0, –5) and has a gradient of 2.

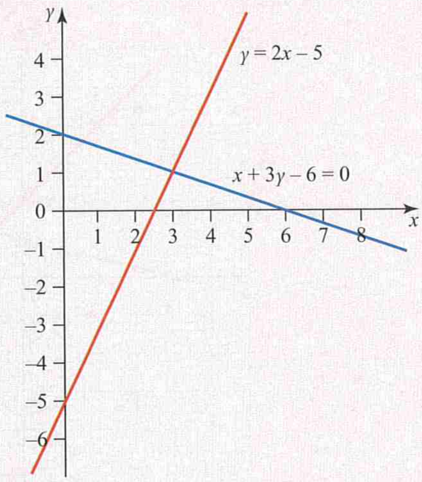


Figure 5.9

(ii) $x + 3y - 6 = 0 \Rightarrow 2x + 6y - 12 = 0$ (multiplying by 2) ①

$y = 2x - 5 \Rightarrow 2x - y - 5 = 0$ ②

① – ② $\Rightarrow 7y - 7 = 0$

$\Rightarrow y = 1$

Substituting $y = 1$ in ① gives $2x + 6 - 12 = 0$

$\Rightarrow x = 3$

The coordinates of the point of intersection are therefore (3, 1).

Discussion point

➜ Graphical methods such as this will have limited accuracy. What factors would affect the accuracy of your solution in this case?

An alternative method for solving these equations simultaneously would be to plot both lines on graph paper and read off the coordinates of the point of intersection.

Example 5.7

(i) Plot the lines $x + y - 2 = 0$ and $4y - x = 4$ on the same set of axes, for $-4 \leqslant x \leqslant 4$, using 1 cm to represent 1 unit on both axes.

(ii) Read off the solution to the simultaneous equations

$x + y - 2 = 0$

$4y - x = 4$

Solution

(i) For each line choose three values of x and calculate the corresponding values of y. Then plot the lines and read off the coordinates of the point of intersection.

$x + y - 2 = 0$

x	–2	0	2
y	4	2	0

$4y - x = 4$

x	–4	0	4
y	0	1	2

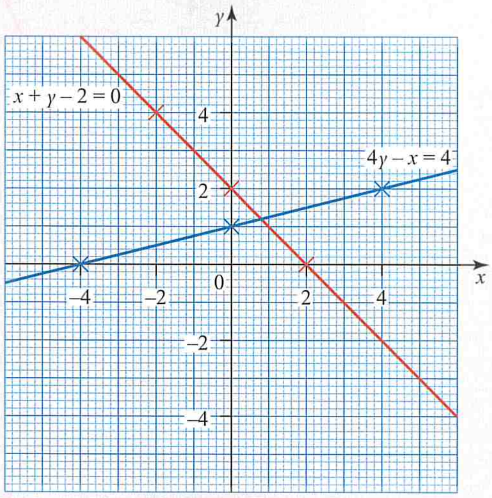


Figure 5.10

(ii) The point of intersection is (0.8, 1.2), so the solution to the simultaneous equations is

$x = 0.8, y = 1.2$

Discussion points

- Why should you plot three points for each line?
- Two lines may not intersect. When is this the case?

Exercise 5C

You will need graph paper for this exercise.

① Solve these pairs of simultaneous equations by plotting their graphs. In each case you are given a suitable range of values of x.

(i) $x = 3y + 1$ $\quad y = x - 1$ $\quad 0 \leqslant x \leqslant 3$

(ii) $3x + 2y = 5$ $\quad x + y = 3$ $\quad -2 \leqslant x \leqslant 2$

② Solve these pairs of simultaneous equations by plotting their graphs. In each case you are given a suitable range of values of x.

(i) $y = 2x - 4$ $\quad 3x + 4y = 17$ $\quad 0 \leqslant x \leqslant 6$

(ii) $6x + y = 1$ $\quad 4x - y = 4$ $\quad 0 \leqslant x \leqslant 2$

③ (i) Plot the lines $x = 4$, $y = x + 4$ and $4x + 3y = 12$ on the same axes for $-1 \leqslant x \leqslant 5$

(ii) State the coordinates of the three points of intersection, and for each point give the pair of simultaneous equations that are satisfied there.

(iii) Work out the area of the triangle enclosed by the three lines.

④ (i) Using the same scale for both axes, plot the lines $2y + x = 4$ and $2y + x = 10$ on the same axes for $0 \leqslant x \leqslant 6$, and say what you notice about them. Why is this the case?

(ii) Add the line $y = 2x$ to your graph. What do you notice now? Can you justify what you see?

(iii) State the coordinates of the two points of intersection, and for each point give the pair of simultaneous equations that are satisfied there.

⑤ A triangle has vertices A(0, 3), B(3, 6) and C(3, 0).

(i) Work out the lengths of the sides of the triangle ABC.

(ii) Work out the equations of the sides of the triangle ABC.

(iii) Describe the triangle ABC.

⑥ A(1, 2), B(2, 5), C(5, 4) and D(4, 1) are the vertices of a quadrilateral ABCD.

(i) Work out the gradients of the sides of the quadrilateral and state two pieces of information that this gives you.

(ii) Work out the lengths of AB and BC.

(iii) What type of quadrilateral is ABCD?

⑦ Alpha and Beta are two rival taxi firms which have the following price structures:

Alpha: A fixed charge of £2 plus 60p per mile.

Beta: A fixed charge of £3 plus 40p per mile.

(i) On the same axes sketch the graph of price (vertical axis) against distance travelled (horizontal axis) for each firm.

(ii) Write down the equation of each line.

(iii) Which firm would you use for a distance of 7 miles?

(iv) For what distance do both firms charge the same?

⑧ When the market price £p of an article varies, so does the number demanded, D and the number supplied, S.

In one case $D = 15 + 0.5p$ and $S = p - 10$

(i) Sketch both lines on the same graph with D and S both on the vertical axis.

The equilibrium position for the market is when the supply and the demand are equal.

(ii) Work out the equilibrium price and the number bought and sold in equilibrium.

6 Dividing a line in a given ratio

You can use similar triangles to work out the coordinates of a point that divides a line in a given ratio if you know the coordinates of the end points of the line.

Look at Figure 5.11. A is (4, 7) and B is (19, 27).

C divides line AB in the ratio 2 : 3, i.e. AC : CB = 2 : 3

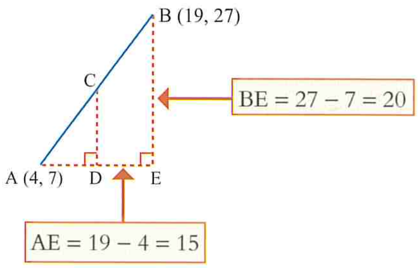


Figure 5.11

$$AC = \frac{2}{5}AB \quad \text{so} \quad AD = \frac{2}{5}AE$$
$$= \frac{2}{5}(15)$$
$$= 6$$

$$\text{Also} \quad CD = \frac{2}{5}BE$$
$$= \frac{2}{5}(20)$$
$$= 8$$

The x-coordinate of C is x-coordinate of A + AD i.e. $4 + 6 = 10$

The y-coordinate of C is y-coordinate of A + CD i.e. $7 + 8 = 15$

Point C is (10, 15).

This result can be generalised.

If C divides line AB in the ratio $p : q$ where A is (x_1, y_1) and B is (x_2, y_2) then

$$\text{C is } \left(\frac{qx_1 + px_2}{p + q}, \frac{qy_1 + py_2}{p + q}\right)$$

ACTIVITY 5.2

Show that the generalisation given on the right is true. Use Figure 5.12. A is (x_1, y_1) and B is (x_2, y_2). C divides line AB in the ratio $p : q$.

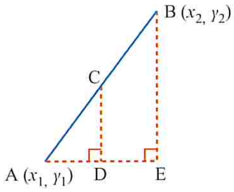


Figure 5.12

Example 5.8

AB is a straight line. A is (−3, 2) and B is (4, −19).

C is a point on AB such that AC : CB is 4 : 3

Work out the coordinates of C.

Solution

Taking (−3, 2) as (x_1, y_1) and (4, −19) as (x_2, y_2) and $p : q$ as 4 : 3

$$x\text{-coordinate of C is } \frac{3(-3) + 4(4)}{3 + 4} = \frac{-9 + 16}{7}$$
$$= 1$$

$$y\text{-coordinate of C is } \frac{3(2) + 4(-19)}{3 + 4} = \frac{6 - 76}{7}$$
$$= -10$$

C is (1, −10).

Example 5.9

In Figure 5.13, PQR is a straight line and PQ : QR is 2 : 5

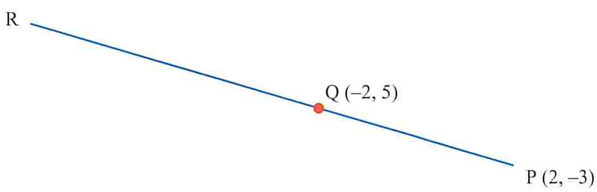


Figure 5.13

Work out the coordinates of R.

Solution

Take (2, –3) as (x_1, y_1) and R as (x_2, y_2). To work out the x-coordinate

$$-2 = \frac{5(2) + 2(x_2)}{2 + 5}$$

$$-2 = \frac{10 + 2x_2}{7}$$

$$-14 = 10 + 2x_2 \quad \text{(multiplying by 7)}$$

$$-24 = 2x_2 \quad \text{(subtracting 10)}$$

$$-12 = x_2$$

To work out the y-coordinate

$$5 = \frac{5(-3) + 2(y_2)}{2 + 5}$$

$$5 = \frac{-15 + 2y_2}{7}$$

$$35 = -15 + 2y_2 \quad \text{(multiplying by 7)}$$

$$50 = 2y_2 \quad \text{(adding 15)}$$

$$25 = y_2$$

R is (–12, 25).

Two shapes are said to be **similar** if corresponding sides are in the same ratio.

For example, the two triangles below are similar:

Discussion points

- What is the ratio of the lengths of the corresponding sides in Figure 5.14?
- What is the ratio of the areas of the two triangles

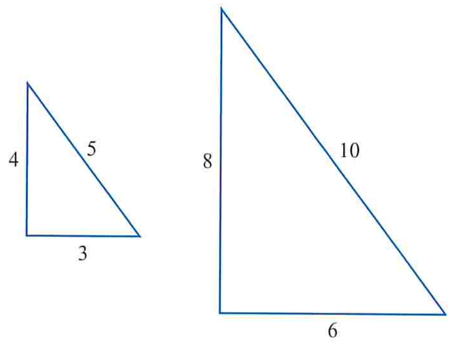


Figure 5.14

! Remember that when increasing or decreasing a length using a scale factor of n, corresponding areas will be increased using a scale factor of n^2.

Exercise 5D

① In each part, AB is a straight line and C is a point on AB. Work out the coordinates of C.

(i)	A is (8, 3)	B is (3, 18)	AC : CB is 3 : 2
(ii)	A is (12, –1)	B is (3, 5)	AC : CB is 1 : 2
(iii)	A is (–2, 4)	B is (14, –4)	AC : CB is 3 : 5
(iv)	A is (11, 9)	B is (–1, 19)	AC : CB is 4 : 1
(v)	A is (0, –6)	B is (–18, –15)	AC : CB is 5 : 4

② In each part, DEF is a straight line.

(i) D is (4, 3) E is (8, 5) DE : EF is 2 : 3 Work out the coordinates of F.

(ii) D is (19, −5) E is (7, 3) DE : EF is 4 : 3 Work out the coordinates of F.

(iii) E is (4, 9) F is (16, 33) DE : EF is 1 : 4 Work out the coordinates of D.

(iv) E is (2, −8) F is (7, −19) DE : EF is 3 : 5 Work out the coordinates of D.

(v) D is (−15, −8) E is (−3, −2) DE : EF is 6 : 5 Work out the coordinates of F.

③ ABC is a straight line. AB is 25% longer than BC.

(i) Work out the ratio AB : BC in its simplest form.

(ii) Work out the coordinates of C.

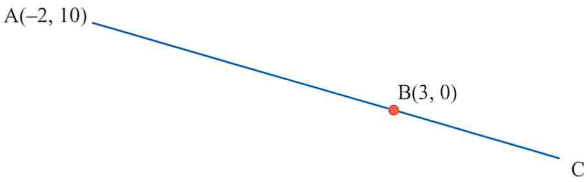


Figure 5.15

④ PRQ is a straight line.

PQ = 4PR

Work out the coordinates of R.

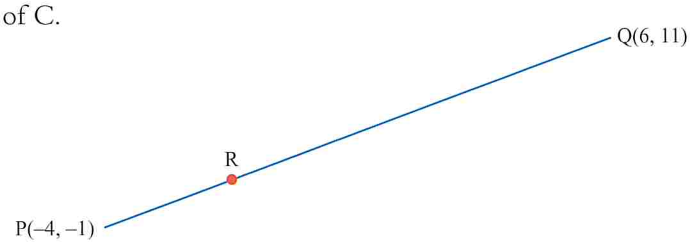


Figure 5.16

⑤ ABC is a straight line.

AC : BC = 8 : 5

Work out the coordinates of B.

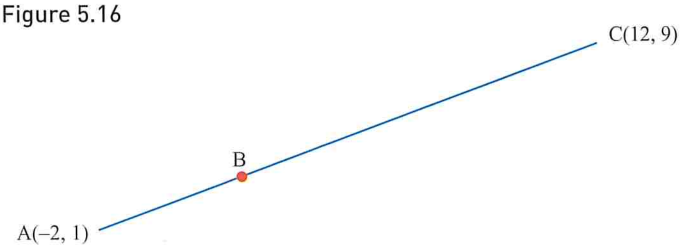


Figure 5.17

⑥ A very important application of ratios is when an architect is drawing a plan for a new building. This will require a scale drawing of the building together with a plan for each room. The plan is for the bedroom plus ensuite bathroom. His drawing uses a scale of 1 : 50.

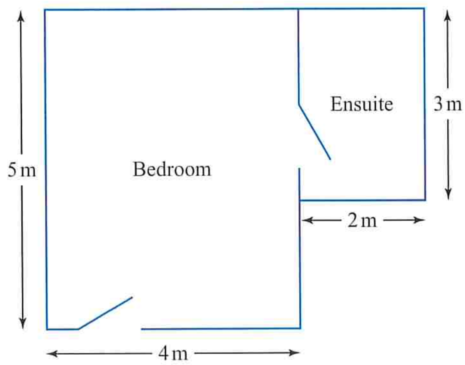


Figure 5.18

(i) What are the dimensions of the bedroom and the ensuite on the scale drawing.

(ii) What is the ratio of the area of the actual rooms to the area on the scale drawing?

⑦ The size of a computer monitor or TV screen is quoted by giving the length of the diagonal across the screen.

(i) My computer monitor has a rectangular screen with dimensions 50 cm by 30 cm. To the nearest cm, what size would this be quoted as?

(ii) My television has a 100 cm screen which is a rectangle with sides in the ratio 33 : 20. What is the height and width of the TV screen to the nearest cm?

⑧ Triangle ABC has a right angle at B and the lengths of the sides are shown in the diagram. A triangle A′B′C′ is inscribed inside ABC so that A′ is the midpoint of BC, B′ is the midpoint of CA and C′ is the midpoint of AB.

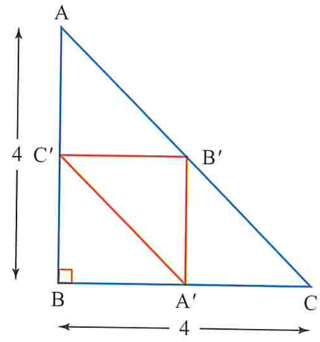


Figure 5.19

(i) Calculate the length of AC.

(ii) Show that triangle A′B′C′ is also a right-angled triangle and calculate the lengths of its sides.

(iii) What do you notice about the ratio of the sides of the smaller triangle to the larger one?

(iv) What do you notice about the ratio of the area of the smaller triangle to the larger one?

7 Equation of a circle

When you draw a circle, you open your compasses to a fixed distance (the radius) and choose a position (the centre) for the point of your compasses. These facts are used to derive the *equation* of the circle.

Circles with centre (0, 0)

Figure 5.20 shows a circle with centre O(0, 0) and radius 4. P(x, y) is a general point on the circle.

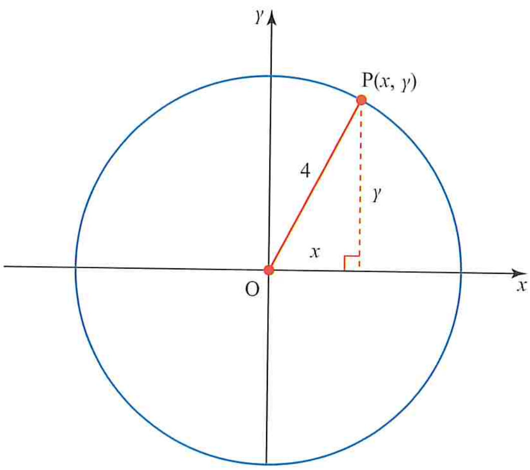


Figure 5.20

Using Pythagoras' theorem, OP $= \sqrt{x^2 + y^2} = 4$

This simplifies to $x^2 + y^2 = 16$, which is the equation of the circle.

This can be generalised. A circle with centre (0, 0), radius r has equation

$$x^2 + y^2 = r^2.$$

Circles with centre (*a*, *b*)

Figure 5.21 shows a circle with centre C(4, 5) and radius 3. P(x, y) is a general point on the circle.

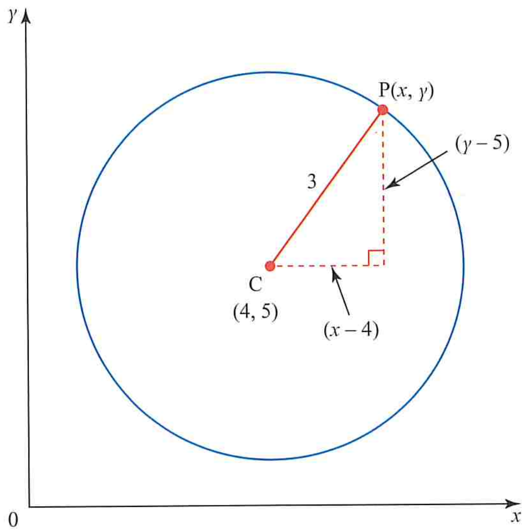


Figure 5.21

$$\text{CP} = \sqrt{(x-4)^2 + (y-5)^2} = 3$$

This simplifies to $(x - 4)^2 + (y - 5)^2 = 9$, which is the equation of the circle.

This can be generalised. A circle with centre (a, b), radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

Note

Multiplying out this equation gives

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2.$$

This rearranges to

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

This form of the equation highlights some of the important characteristics of the equation of a circle. In particular

- the coefficients of x^2 and y^2 are equal
- there is no xy term.

Example 5.10

Write down the centre and radius of the circle

$$x^2 + (y + 3)^2 = 25$$

Solution

Comparing with the general equation for a circle with radius r and centre (a, b),

$$(x - a)^2 + (y - b)^2 = r^2$$

gives $a = 0$, $b = -3$ and $r = 5$

$\Rightarrow$ the centre is $(0, -3)$, the radius is 5

Example 5.11

Figure 5.22 shows a circle with centre $(1, -2)$, which passes through the point $(4, 2)$.

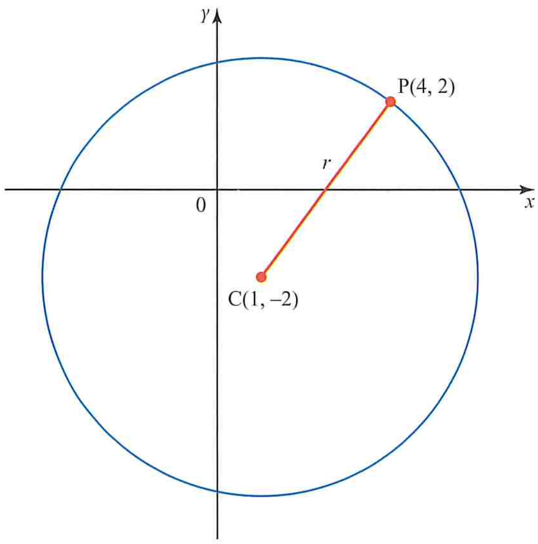


Figure 5.22

(i) Work out the radius of the circle.

(ii) Write down the equation of the circle.

Solution

(i) Use the two points you are given to work out the radius of the circle.

$$r^2 = (4 - 1)^2 + (2 - (-2))^2$$
$$= 25$$
$$\Rightarrow \quad \text{radius} = 5$$

(ii) Now using $(x - a)^2 + (y - b)^2 = r^2$

$$\Rightarrow \quad (x - 1)^2 + (y + 2)^2 = 25$$

is the equation of the circle.

Equation of a circle

Example 5.12

Show that the equation $x^2 + y^2 + 4x - 6y - 3 = 0$ represents a circle.

Hence give the coordinates of the centre and the radius of the circle.

Solution

Using completing the square

$$x^2 + 4x + y^2 - 6y = 3$$

$$\Rightarrow \quad x^2 + 4x + 4 + y^2 - 6y + 9 = 3 + 4 + 9$$

$$\Rightarrow \quad (x+2)^2 + (y-3)^2 = 3 + 4 + 9$$

$$\Rightarrow \quad (x+2)^2 + (y-3)^2 = 16$$

This represents a circle with centre $(-2, 3)$, radius 4

Exercise 5E

① Write down the equations of these circles.

(i) centre $(1, 2)$, radius 3

(ii) centre $(4, -3)$, radius 4

(iii) centre $(1, 0)$, radius 5

(iv) centre $(-2, -2)$, radius 2

(v) centre $(-4, 3)$, radius 1

② For each of the circles given below

(a) state the coordinates of the centre

(b) state the radius

(c) sketch the circle, paying particular attention to its position in relation to the origin and the coordinate axes.

(i) $x^2 + y^2 = 25$

(ii) $(x - 3)^2 + y^2 = 9$

(iii) $(x + 4)^2 + (y - 3)^2 = 25$

(iv) $(x + 1)^2 + (y + 6)^2 = 36$

(v) $(x - 4)^2 + (y - 4)^2 = 16$

③ Work out the equation of the circle with centre $(2, -3)$ which passes through $(1, -1)$.

④ A and B are $(4, -4)$ and $(2, 6)$ respectively. Work out

(i) the midpoint C of AB

(ii) the distance AC

(iii) the equation of the circle that has AB as its diameter.

⑤ Show that the equation $x^2 + y^2 - 4x - 8y + 4 = 0$ represents a circle.

Hence give the coordinates of the centre and the radius of the circle, and sketch the circle.

⑥ Why does the equation $x^2 - 4x + y^2 - 4y + 22 = 0$ not represent a circle?

⑦ A circle of radius 5 cm passes through the points $(0, 0)$ and $(0, 6)$.

Sketch two possible positions of the circle and write down the equation in each case.

⑧ $(6, 3)$ is a point on the circle with centre $(11, 8)$.

(i) Work out the radius of the circle.

(ii) Work out the equation of the circle.

(iii) Work out the coordinates of the other point where the diameter through $(6, 3)$ meets the circle.

Prior knowledge

Students met a number of circle facts at GCSE and a reminder of some of these is given below. See section 6.4 on circle theorems.

Circle geometry facts

The angle in a semi-circle is 90°

An alternative way of expressing this is to say that the angle subtended by the diameter at any point on the circumference is 90°.

AB is a diameter.

P is a point on the circumference.

Angle APB = 90°.

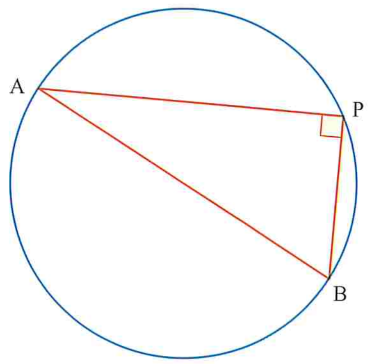


Figure 5.23

Discussion point

- Can you prove RM = MS by using congruent triangles?

The perpendicular from the centre to a chord bisects the chord

C is the centre.

RS is a chord.

M is the midpoint of RS.

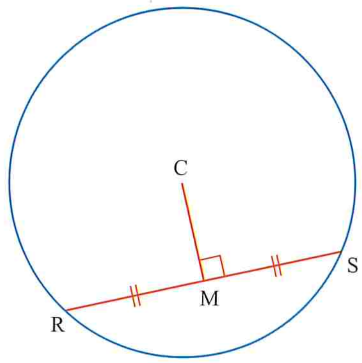


Figure 5.24

The angle between tangent and radius is 90°

C is the centre.

TQ is a tangent, touching the circle at Q.

QC is a radius.

Angle TQC = 90°.

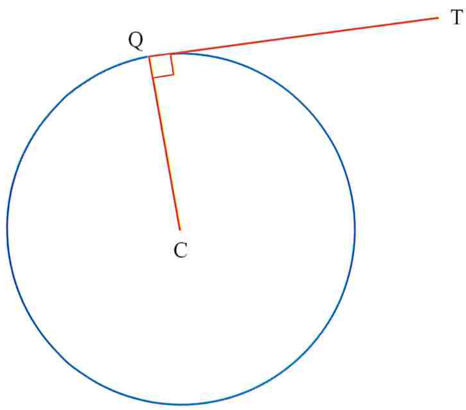


Figure 5.25

Two tangents from a point to a circle are equal in length

> **Discussion point**
>
> → Can you prove TA = TB by using congruent triangles?

From any point outside a circle it is possible to draw two tangents to that circle.

C is the centre.

TA and TB are tangents, touching the circle at A and B.

AC and BC are radii.

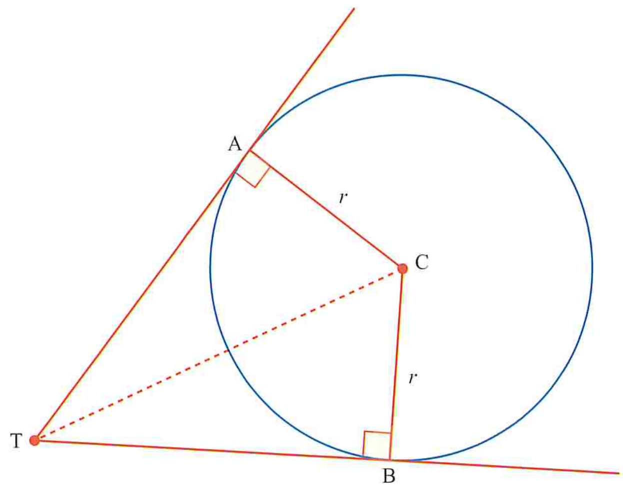


Figure 5.26

Example 5.13

The circle in Figure 5.27 has centre C.

PT is a tangent that touches the circle at P.

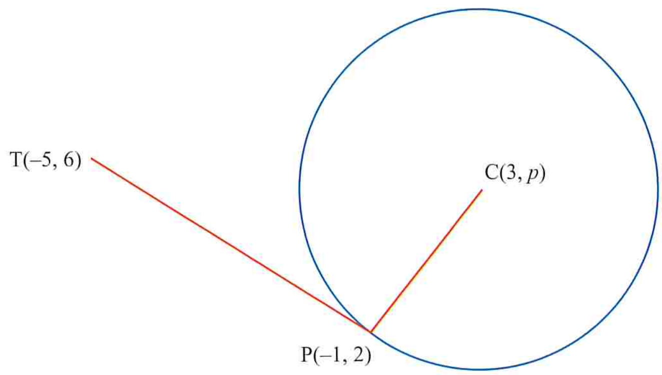


Figure 5.27

Work out the value of p.

Solution

Work out gradient of PT.

Gradient of PT is

$$\frac{2-6}{-1-(-5)} = \frac{-4}{-1+5}$$
$$= \frac{-4}{4}$$
$$= -1$$

Lines PT and PC are perpendicular since the angle between tangent and radius is 90°, so gradient of PC is 1.

$$\frac{p-2}{3-(-1)} = 1$$

$$\frac{p-2}{4} = 1$$

$$p - 2 = 4$$

$$p = 6$$

Example 5.14

A circle has centre C and passes through A(–2, 1) and B(4, 2).

Work out the equation of the line that is perpendicular to AB and passes through C.

Give your answer in the form $ax + by = c$.

Solution

Sketch the circle.

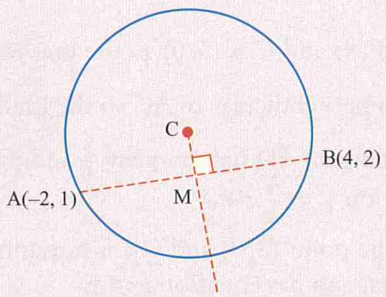


Figure 5.28

The perpendicular from the centre to a chord bisects the chord, so the required line will pass through the midpoint M of the chord AB.

Coordinates of M are $\left(\frac{-2+4}{2}, \frac{1+2}{2}\right) = \left(1, \frac{3}{2}\right)$

Gradient of AB is $\frac{2-1}{4-(-2)} = \frac{1}{6}$

Lines CM and AB are perpendicular.

Using $m_1 m_2 = -1$

Gradient of required line is –6.

Using $y - y_1 = m(x - x_1)$

$$y - \frac{3}{2} = -6(x - 1)$$

The question asks for the answer in the form $ax + by = c$.

$$y - \frac{3}{2} = -6x + 6$$

$$2y - 3 = -12x + 12$$

Multiply both sides by 2.

$$12x + 2y = 15$$

Example 5.15

(i) Using the fact that a tangent is perpendicular to the radius passing through the point of contact, work out the equation of the tangent to the circle $(x-3)^2 + y^2 = 25$ at the point A(0, 4).

(ii) Work out the coordinates of the point D where this tangent intersects the tangent to the circle through the point B(8, 0).

(iii) Show that these two tangents are equal in length.

Solution

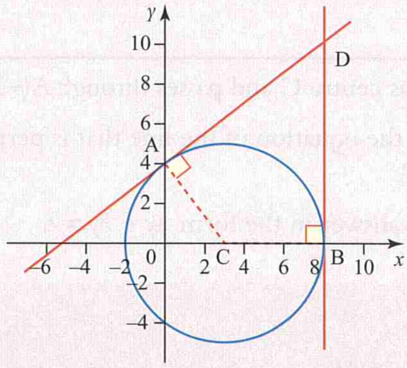


Figure 5.29

(i) A is (0, 4) and C is (3, 0) so the gradient of AC $= -\frac{4}{3}$.

AD is perpendicular to AC so the gradient of AD is $+\frac{3}{4}$.

The tangent AD has gradient $\frac{3}{4}$ and passes through (0, 4) so has equation $y = \frac{3}{4}x + 4$

(ii) B is the point (8, 0) and CB is horizontal, so the tangent at B is vertical and has equation $x = 8$

Solving $y = \frac{3}{4}x + 4$ and $x = 8$ simultaneously gives

$y = \frac{3}{4}(8) + 4 = 10$, so D is the point (8, 10).

(iii) A(0, 4) and D(8, 10) $\Rightarrow$ AD $= \sqrt{(8-0)^2 + (10-4)^2} = 10$

B(8, 0) and D(8, 10) $\Rightarrow$ BD = 10 so the two tangents are equal in length.

Exercise 5F

If a diagram is not given, drawing a sketch may help.

① AB is a diameter of a circle. P is a point on the circumference of the circle.
A is (2, 8) and P is (4, −2).
Work out the gradient of BP.

② A circle has centre C.
RS is a chord of the circle and R is (−1, 6).
Y(2, 3) is a point on RS such that angle CYR = 90°.
Work out the coordinates of S.

③ Look at Figure 5.30.

AB is a diameter of the circle.

A is $(k, 5)$, P is $(3, 8)$ and B is $(7, 2)$.

Work out the value of k.

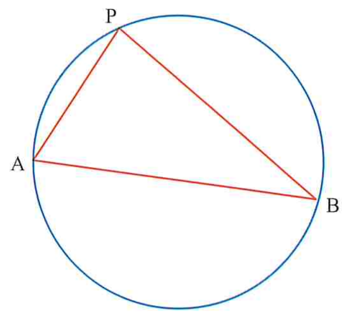


Figure 5.30

④ Figure 5.31 shows a circle, centre C.

AB is a chord of a circle.

D is a point on AB such that angle ADC is 90°.

Work out the equation of the line that passes through A and C.

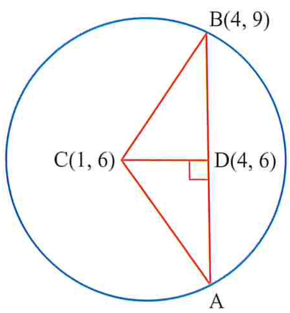


Figure 5.31

⑤ Look at Figure 5.32. T$(3, -4)$ is a point on the circumference of the circle $x^2 + y^2 = 25$

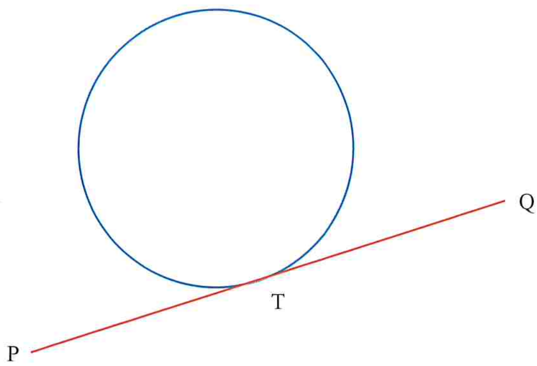


Figure 5.32

Work out the equation of the tangent PTQ.

Give your answer in the form $y = mx + c$.

⑥ Figure 5.33 shows a circle that intersects the x-axis at $(-2, 0)$ and $(6, 0)$.

The centre of the circle is $(a, 3)$.

Work out the equation of the circle.

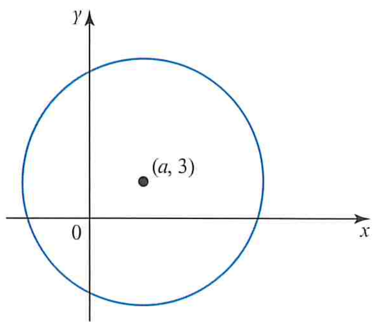


Figure 5.33

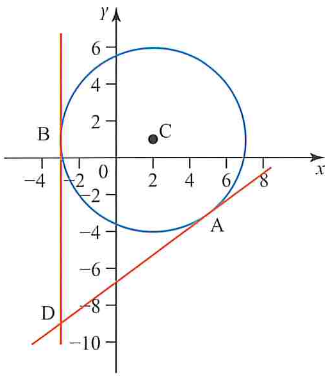


Figure 5.34

⑦ Figure 5.34 shows a circle with centre C(2, 1) and radius 5 units together with the tangent at the point A(5, –3).

(i) Write down the equation of the circle.

(ii) Work out the equation of the tangent and verify that the point D(–3, –9) lies on the tangent.

(iii) Write down the equation of the other tangent through the point D(–3, –9) and state the coordinates of the point B where this touches the circle.

(iv) Work out the lengths of the two tangents from D to the circle.

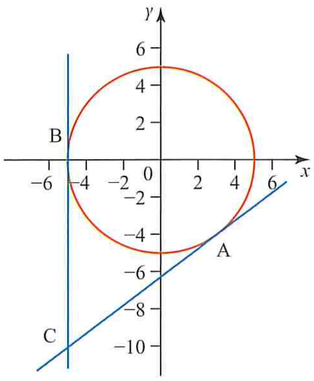


Figure 5.35

⑧ Figure 5.35 shows a circle $x^2 + y^2 = 25$ together with the tangent to the circle at the point A(3, –4).

(i) Show that the tangent at A has equation $3x - 4y - 25 = 0$

(ii) Write down the equation of the tangent to the circle at the point B(–5, 0).

(iii) Work out the coordinates of the point of intersection, C, of these tangents.

(iv) Show, by calculation, that CA = CB.

FUTURE USES

- At A-Level Maths and Further Maths you will do more work on circles and also study hyperbolas, parametric equations and polar coordinates.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- identify parallel and perpendicular lines
- work out the equation of a line given the gradient of a line parallel or perpendicular to it and at least one point that it passes through
- calculate the distance between two points
- work out the coordinates of the midpoint of a line joining two points
- recognise different forms for the equation of a straight line
- find the point of intersection of two lines
- divide a straight line in a given ratio
- recognise the equation of a circle
- write down the equation of a circle given the radius and the coordinates of the centre
- solve a variety of problems involving tangents to circles.

KEY POINTS

1 Two lines are parallel when their gradients are equal.

2 Two lines are perpendicular when the product of their gradients is -1.

3 When the points A and B have coordinates (x_1, y_1) and (x_2, y_2) respectively then

$$\text{distance AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{midpoint of AB is } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

4 The coordinates of the point of intersection of two lines are found by solving their equations simultaneously.

5 If C divides line AB in the ratio $p:q$ where A is (x_1, y_1) and B is (x_2, y_2) then

$$\text{C is } \left(\frac{qx_1 + px_2}{p + q}, \frac{qy_1 + py_2}{p + q}\right).$$

6 The equation of a circle with centre (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

When the centre is at the origin $(0, 0)$ this simplifies to

$$x^2 + y^2 = r^2.$$

7 Circle facts:

- The angle in a semi-circle is $90°$.
- The perpendicular from the centre to a chord bisects the chord.
- The angle between tangent and radius is $90°$.
- The tangents to a circle from an external point are equal in length.

6 Geometry I

The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds.
John Maynard Keynes

Much of the work in this chapter will already have been covered in your GCSE studies.

Knowledge of geometry topics will be needed within many sections of the specification.

This section provides a summary of the main facts that are required.

1 Mensuration

You need to recall these formulae.

Area of triangle $= \frac{1}{2} \times$ base $\times$ height

Area of parallelogram = base $\times$ height

Area of a trapezium $= \frac{1}{2} \times$ sum of parallel sides $\times$ distance between them

Circumference of circle $= \pi d = 2\pi r$ Area of circle $= \pi r^2$

Volume of prism = area of cross section $\times$ length

These formulae will be given in the examination in the relevant question.

Volume of pyramid $\frac{1}{3} \times$ base area $\times$ height

Volume of cone $= \frac{1}{3}\pi r^2 h$ Curved surface area of cone $= \pi r l$

Volume of sphere $= \frac{4}{3}\pi r^3$ Surface area of sphere $= 4\pi r^2$

ACTIVITY 6.1

Write down the square of all the integers from 1 to 25 inclusive.

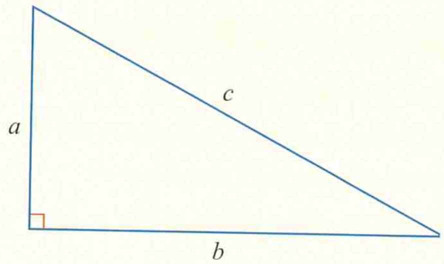
Check that $5^2 = 3^2 + 4^2$.

Write down as many other examples of $c^2 = a^2 + b^2$ as you can find.

How is each set of a, b and c linked to a right-angled triangle?

Prior knowledge

2 Pythagoras' theorem



$c^2 = a^2 + b^2$

Figure 6.1

Pythagorean triples

The following are all Pythagorean triples as each set of three numbers satisfies $c^2 = a^2 + b^2$.

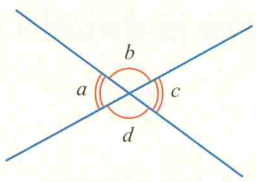
3, 4, 5 5, 12, 13 8, 15, 17 7, 24, 25

Using similar triangles, any multiple or fraction of each set will also be a Pythagorean triple.

For example, 9, 12, 15 2.5, 6, 6.5 16, 30, 34 1.4, 4.8, 5

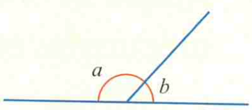
3 Angle facts

- When two lines intersect, there are two angle facts to remember:



$a = c$
$b = d$

Vertically opposite angles are equal



$a + b = 180°$

Adjacent angles on a straight line add up to 180°

Figure 6.2

- Parallel lines have the same gradient (slope). When a third line intersects a pair of parallel lines, there are three angle properties relating to these that you need to remember:

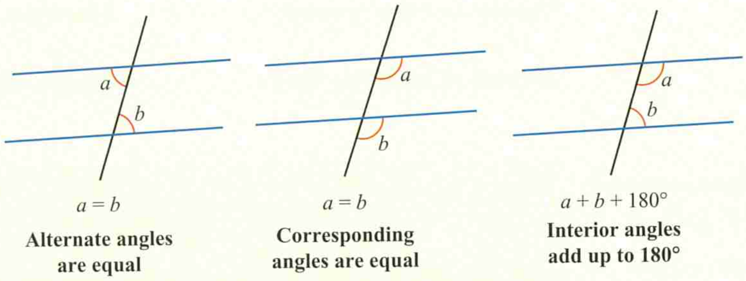


Figure 6.3

Angle properties of polygons

- Angle sum of a triangle is $180°$.
- Angle sum of a quadrilateral (4 sides) is $360°$.
- Angle sum of a pentagon (5 sides) is $540°$
- Angle sum of an n-sided polygon is $(n - 2) \times 180°$.

Special quadrilaterals

- A parallelogram has opposite angles equal.
- A rhombus is a parallelogram with all four sides of equal length.
- A trapezium is a quadrilateral with one pair of parallel sides.
- A kite has one line of symmetry and the diagonals intersect at $90°$.

Special polygons

- A regular polygon is a figure with all sides of equal length and all angles of equal size.

4 Circle theorems

The following angle properties should be known.

Angle at the centre is double the angle at the circumference

C is the centre.

A, B and D are points on the circumference.

Angle ACB = 2 × angle ADB.

Both angles must be subtended from the same arc.

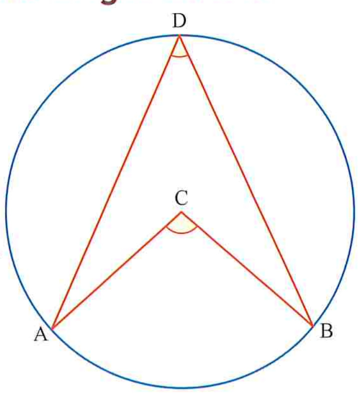


Figure 6.4

S is the centre.

P, Q and R are points on the circumference.

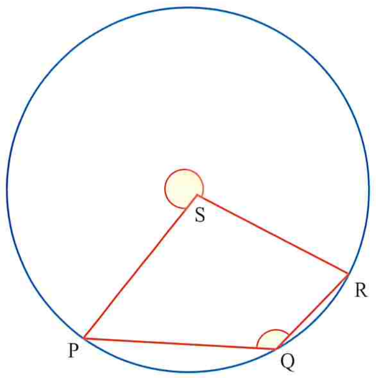


Figure 6.5

Reflex angle PSR = 2 × angle PQR.

Angle in a semi-circle = 90°

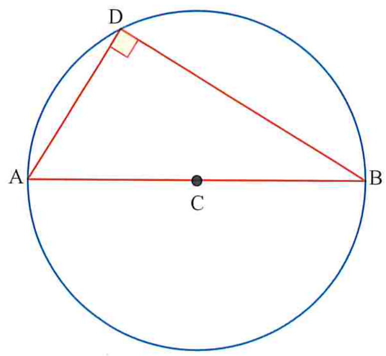
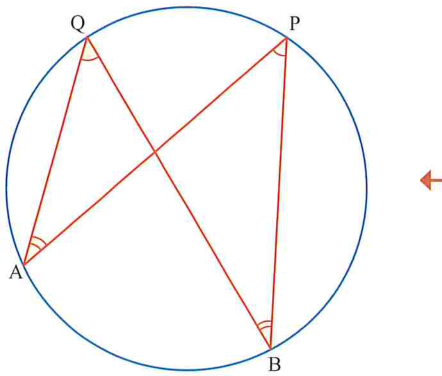


Figure 6.6

Angle ADB is referred to as the angle in a semi-circle and is always 90°.

Angles in the same segment are equal



Can also be referred to as **Angles subtended by the same arc are equal**.

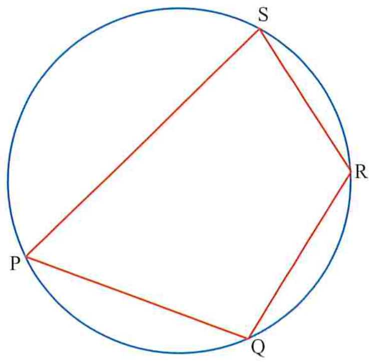
Figure 6.7

A, B, P and Q are points on the circumference.

Angle APB = angle AQB.

Also, angle QAP = angle QBP.

Opposite angles of a cyclic quadrilateral add up to 180°



A cyclic quadrilateral has all four vertices on the circumference.

Figure 6.8

P, Q, R and S are points on the circumference.

Angle PQR + angle RSP = 180°.

Also, angle SPQ + angle QRS = 180°.

Alternate segment theorem

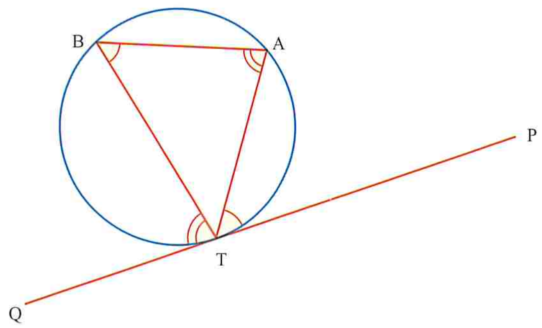


Figure 6.9

PTQ is a tangent, touching the circle at T.

A and B are points on the circumference.

Angle ATP = angle TBA.

Also, Angle QTB = Angle BAT.

The angle between the tangent PT and the chord TA is equal to the angle in the other segment.

The angle between the tangent QT and the chord TB is equal to the angle in the other segment.

Circle theorems and other angle facts will be needed in the Geometric proof section later in this chapter.

Exercise 6A

① Work out angle x and angle y. C is the centre of the circle.

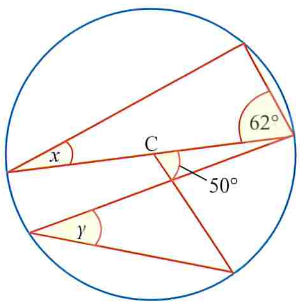


Figure 6.10

② Work out angle x. C is the centre of the circle.

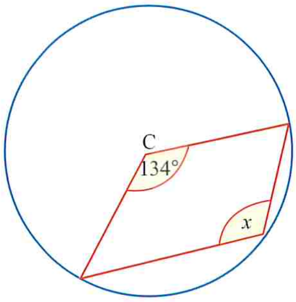


Figure 6.11

③ Work out angle x and angle y. QTP is a tangent.

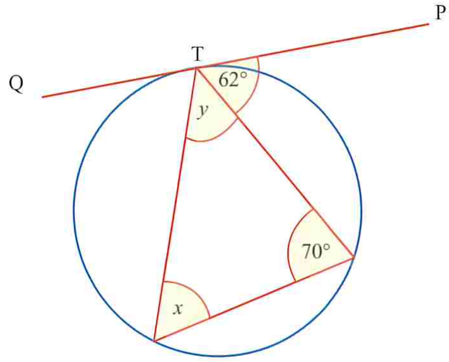


Figure 6.12

④ Work out angle x and angle y. C is the centre of the circle.

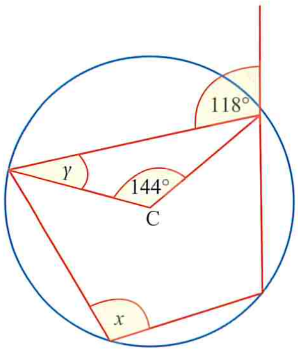


Figure 6.13

⑤ Work out angle x. C is the centre of the circle.

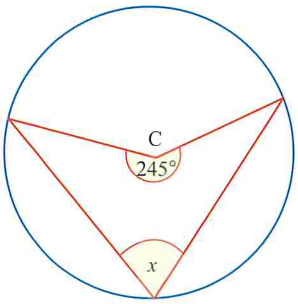


Figure 6.14

⑥ PTQ is a tangent. Work out angle x.

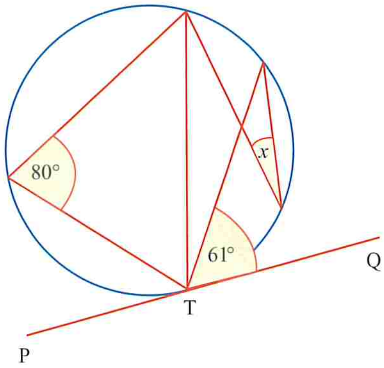


Figure 6.15

⑦ Work out angle x. AB is a diameter.

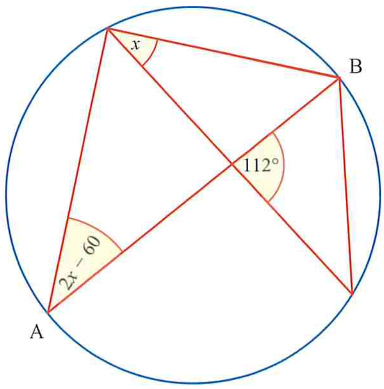


Figure 6.16

⑧ Work out angle c. AB is a tangent.

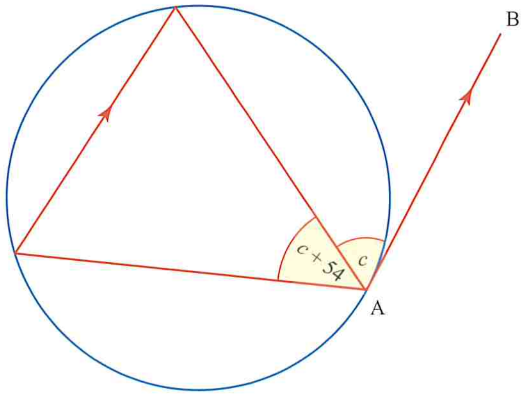


Figure 6.17

⑨ Work out x and y.

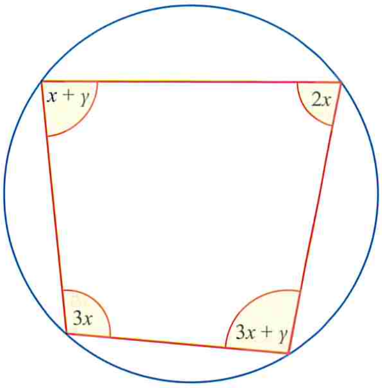


Figure 6.18

5 Geometric proof

In this section you will need to construct formal proofs.

The geometrical properties used must be stated using correct notation and vocabulary.

Example 6.1

In triangle BCD, BC = BD.

ABC is a straight line.

Prove that angle ABD = $2x$.

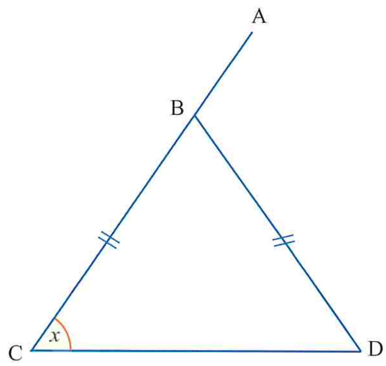


Figure 6.19

Solution

Abbreviations may be used but the reasons must be unambiguous.

Method 1

angle CDB = x	(base angles of isosceles triangle)
angle CBD = $180 - 2x$	(angle sum of triangle)
angle ABD = $2x$	(adjacent angles on a straight line)

Method 2

angle CDB = x	(base angles of isosceles triangle)
angle ABD = $2x$	(exterior angle of triangle = sum of interior opposite angles)

There will often be more than one method in a geometric proof. You only need to provide one.

Example 6.2

AP is a tangent that touches the circle at P.

AP is parallel to QR.

Prove that triangle PQR is isosceles.

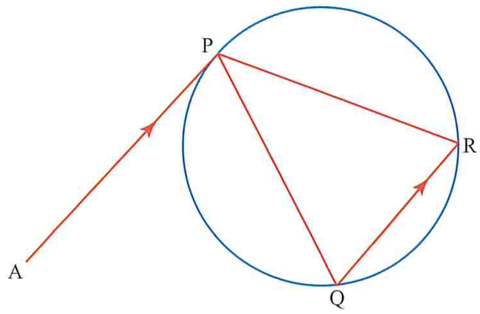


Figure 6.20

Solution

Method 1

angle APQ = angle PQR (alternate angles)

angle APQ = angle PRQ (alternate segment theorem)

Therefore, angle PQR = angle PRQ

Triangle with two equal angles is isosceles.

Method 2

This is very similar to method 1 but starts by introducing a lower case letter.

angle APQ = x

angle PQR = x (alternate angles)

angle PRQ = x (alternate segment theorem)

Therefore, angle PQR = angle PRQ

Triangle with two equal angles is isosceles.

Example 6.3

PQRS is a cyclic quadrilateral.

C is the centre.

Angle QPS = y

Angle QCR = $2x$

Angle SQR = 40°

Prove that $y = x + 40$

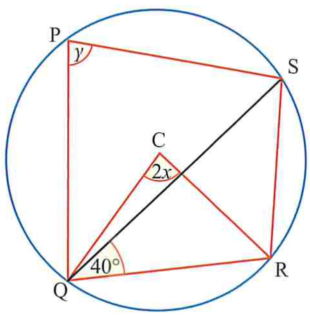


Figure 6.21

Note

The use of congruent triangles will not be required.

The examples show the recommended way to present a proof.

Solution

angle QSR = x (angle at circumference is half angle at centre)

angle SRQ = $180 - y$ (opposite angles of cyclic quadrilateral)

In triangle QRS, $x + 180 - y + 40 = 180$ (angle sum of triangle)

Rearranging $x + 40 = y$

Exercise 6B

① AC is a diameter. B is a point on the circumference.

Prove that $x = 90 - y$.

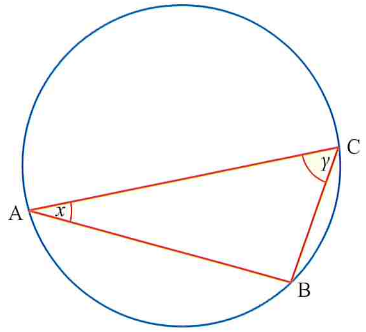


Figure 6.22

② ABCD is parallel to EFG.

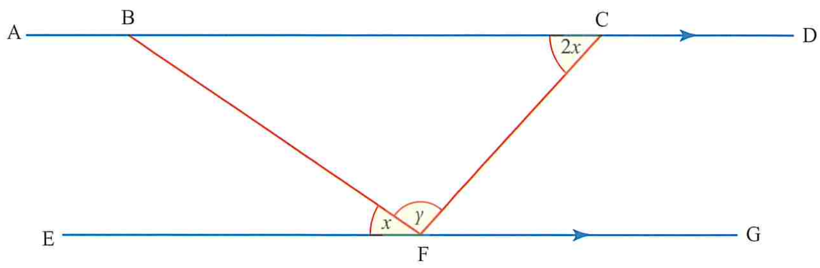


Figure 6.23

Prove that $3x + y = 180$.

③ AB is a diameter. X and Y are points on the circumference.

Prove that $a + b = 90$

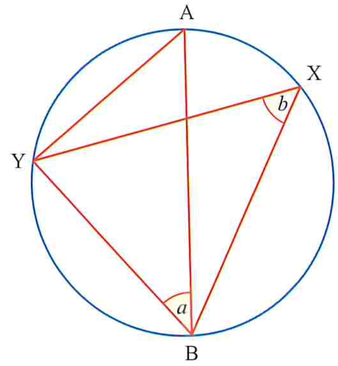


Figure 6.24

④ CBE and DBF are straight lines.

CD is parallel to AB.

BC = BD

Prove that angle ABC = angle ABF.

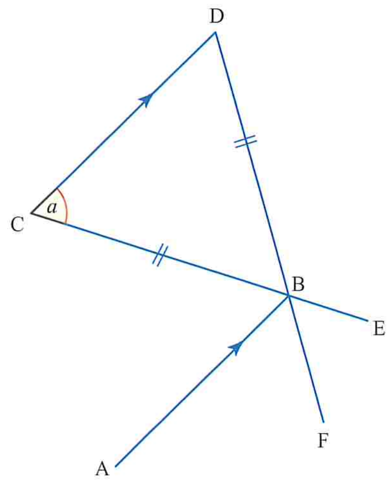


Figure 6.25

⑤ PT is a tangent, touching the circle at T. C is the centre.

M and N are points on the circumference.

Prove that angle TMN = 45 − y.

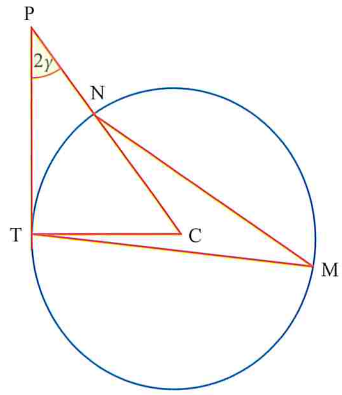


Figure 6.26

⑥ AB is a tangent, touching the circle at B.

ADC is a straight line.

AB = BC

Prove that triangle ABD is isosceles.

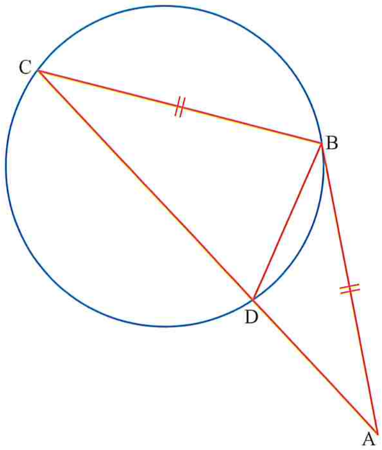


Figure 6.27

⑦ DEFG is a cyclic quadrilateral. C is the centre.

Prove that $x = 2y - 80$

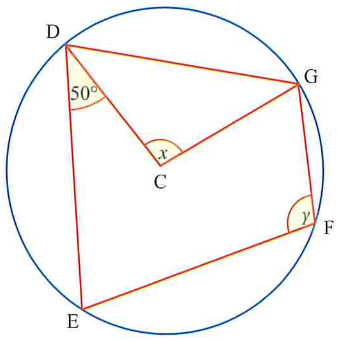


Figure 6.28

⑧ C is the centre.

P, Q and R are points on the circumference with PQ = QR.

Prove that $y = 2x$.

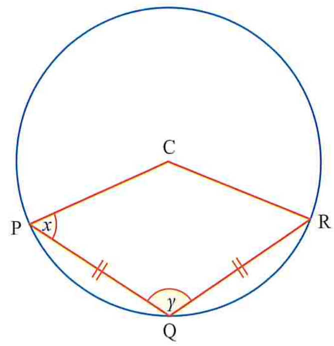


Figure 6.29

6 Trigonometry in two dimensions

You have met definitions of the three trigonometric functions, sin, cos and tan, using the sides of a right-angled triangle.

sin is an abbreviation of sine, cos of cosine and tan of tangent.

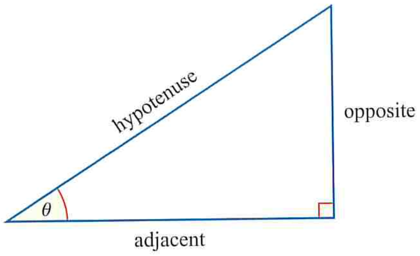


Figure 6.30

Discussion point

➜ Do these definitions work for angles of any size?

In Figure 6.30

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

ACTIVITY 6.2

(i) Using only a pencil, ruler and protractor, estimate $\sin 62°$.

(ii) Use your calculator to check your percentage error.

(iii) Suggest a way of reducing the percentage error when using this method.

Example 6.4

Work out the length of the side marked a in the triangle in Figure 6.31.

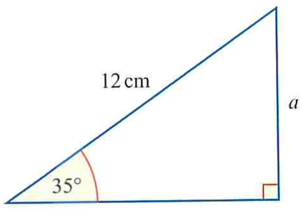


Figure 6.31

Solution

Side a is *opposite* the angle of 35°, and the *hypotenuse* is 12 cm, so we use sin 35°.

$$\sin 35° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{a}{12}$$

$$\Rightarrow \quad a = 12\sin 35°$$

$$\Rightarrow \quad a = 6.9\,\text{cm (1 d.p.)}$$

Example 6.5

RWC

The diagram represents a ladder leaning against a wall.

Work out the length of the ladder.

Give your answer to 3 significant figures.

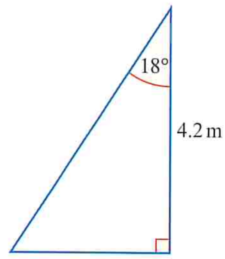


Figure 6.32

Solution

The side of length 4.2 m is *adjacent* to the angle of 18°, and we want the *hypotenuse* so use cos 18°.

$$\cos 18° = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{4.2}{\text{hypotenuse}}$$

$$\text{hypotenuse} = \frac{4.2}{\cos 18°}$$

$$= 4.42\text{ m (3 s.f.)}$$

Example 6.6

Work out the size of the angle marked θ in the triangle in Figure 6.33.

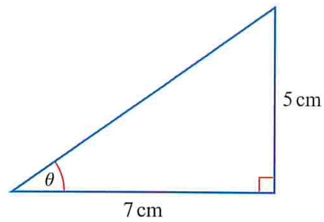


Figure 6.33

Solution

The sides whose lengths are known are those *opposite* and *adjacent* to θ so we use tan θ.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{7}$$

$$\Rightarrow \quad \theta = 35.5° \text{ (1 d.p.)}$$

Discussion point

→ The full calculator value for $\frac{5}{7}$ has been used to work out the value of θ.

What is the least number of decimal places that you could use to give the same value for the angle (to 1d.p.) in this example?

Example 6.7

RWC

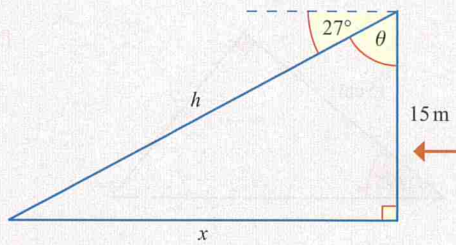
A bird flies straight from the top of a 15 m tall tree, at an angle of depression of 27°, to catch a worm on the ground.

(i) How far does the bird fly?

(ii) How far was the worm from the bottom of the tree?

Solution

First draw a sketch, labelling the information given and using letters to mark what you want to find.



Remember, *angles of depression* are measured down from the horizontal and angles of elevation are measured up from the horizontal.

Figure 6.34

(i) $\theta + 27° = 90°$

$\Rightarrow \quad \theta = 63°$

$\cos 63° = \dfrac{15}{h}$

$\Rightarrow \quad h = \dfrac{15}{\cos 63°} = 33.040\,338\,97$

The bird flies 33 m.

! Make sure that you record the full calculator value of h for future use.

(ii) Using Pythagoras' theorem

$h^2 = x^2 + 15^2$

$\Rightarrow \quad x^2 = 33.040\,338\,97^2 - 15^2 = 866.663\,999$

$\Rightarrow \quad x = 29.439\,157\,58$

The worm is 29.4 m from the bottom of the tree.

Note

Questions involving right-angled triangles will often entail applying trigonometry in a context. Examples and exercises include some questions without a context to provide practice of the skills needed in applications questions.

Discussion point

→ If you used trigonometry for part (ii) of this question, which would be the best function to use? Why?

Historical note

The word for trigonometry is derived from three Greek words.

Tria: *three* gonia: *angle* metron: *measure*

(τρια) (γονια) (μετρον)

This shows how trigonometry developed from studying angles, often in connection with astronomy, although the subject was probably discovered independently by a number of people. Hipparchus (150 BC) is believed to have produced the first trigonometric tables which gave lengths of chords of a circle of unit radius. His work was further developed by Ptolemy in AD100.

Exercise 6C

① Work out the length marked x in each of these triangles. Give your answers correct to 1 decimal place.

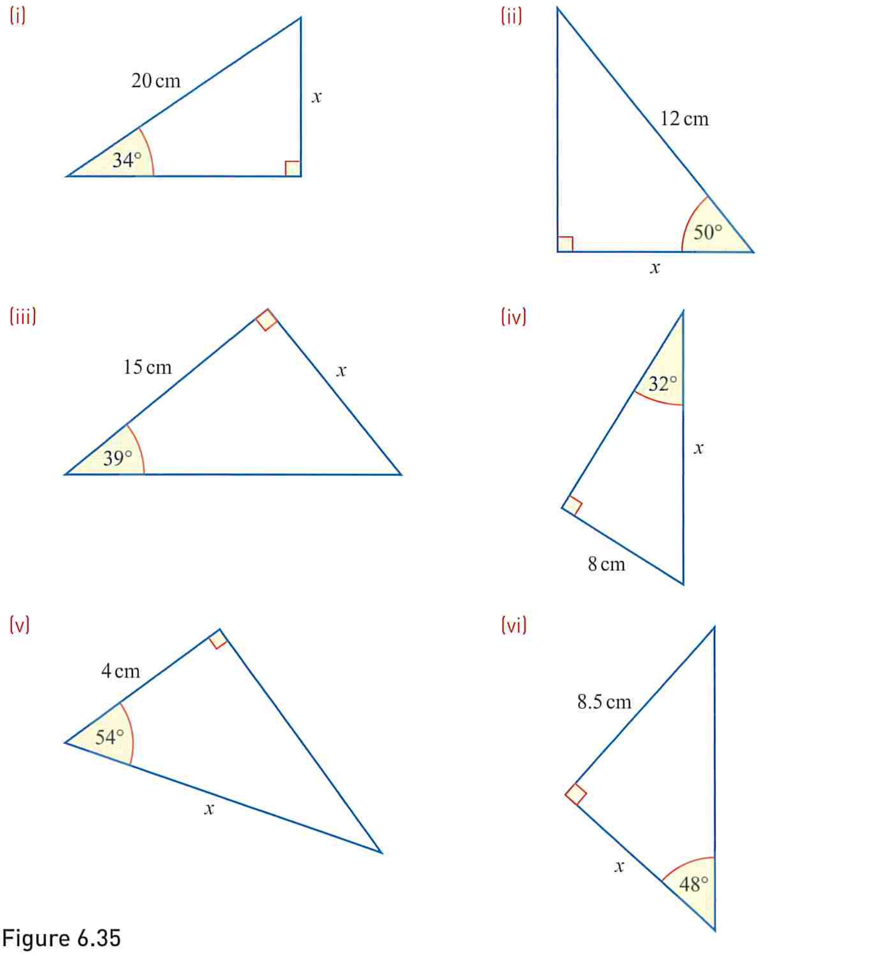
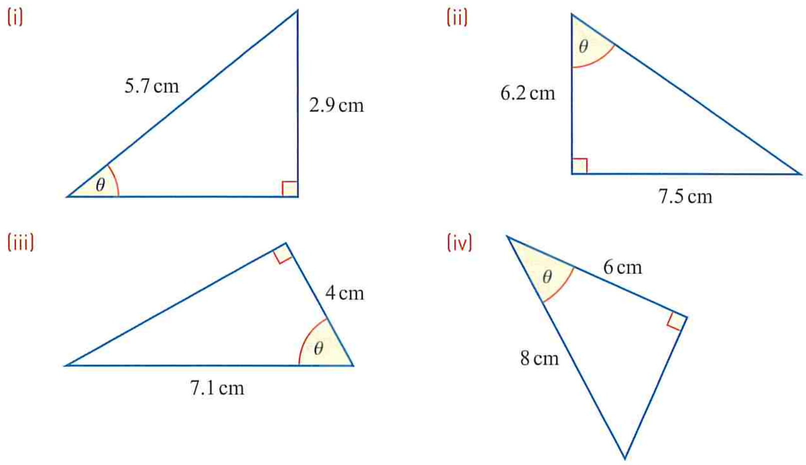


Figure 6.35

② Work out the size of the angle marked θ in each of these triangles. Give your answers correct to 1 decimal place.



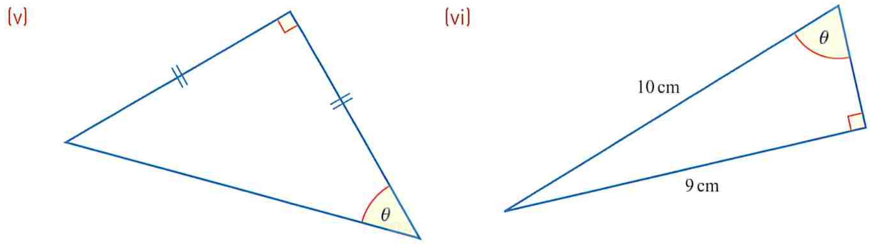


Figure 6.36

③ In an isosceles triangle, the line of symmetry bisects the base of the triangle. Use this fact to work out the angle θ and the lengths x and y in these diagrams.

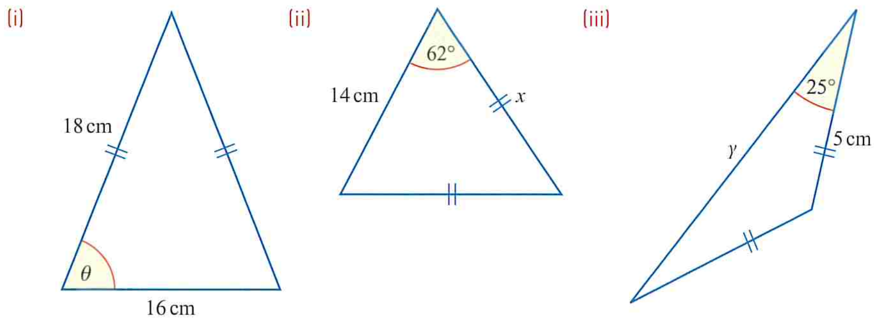


Figure 6.37

RWC ④ A ladder 5 m long rests against a wall. The foot of the ladder makes an angle of 65° with the ground.

How far up the wall does the ladder reach?

RWC ⑤ From the top of a vertical cliff 30 m high, the angle of depression of a boat at sea is 21°.

How far is the boat from the bottom of the cliff?

RWC ⑥ From a point 120 m from the base of an office block, the angle of elevation of the top of the block is 67°.

How tall is the block?

⑦ A rectangle has sides of length 12 cm and 8 cm.

What angle does the diagonal make with the longest side?

RWC ⑧ The diagram shows the positions of three airports:

E (East Midlands), M (Manchester) and L (Leeds).

The distance from M to L is 65 km on a bearing of 060°.

Angle LME = 90° and ME = 100 km.

(i) Calculate, correct to 3 significant figures, the distance LE.

(ii) Calculate, correct to the nearest degree, the size of angle MEL.

(iii) An aircraft leaves M at 10.45 am and flies direct to E, arriving at 11.03 am. Calculate, correct to 3 significant figures, the average speed of the aircraft in kilometres per hour.

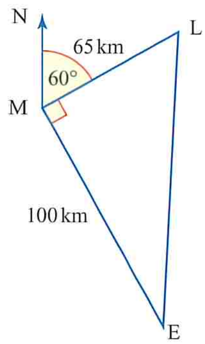


Figure 6.38

Angles of 45°, 30° and 60°

The sine, cosine and tangent of these angles have exact values.

When working without a calculator, the exact values should be known or derived.

Discussion point

→ What would the results be if you used AB = BC = 2 units?

Consider an isosceles right-angled triangle with AB = BC = 1 unit.

Using Pythagoras' theorem

$$AC^2 = 1^2 + 1^2$$
$$AC = \sqrt{2}$$

$$\sin 45° = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = 1$$

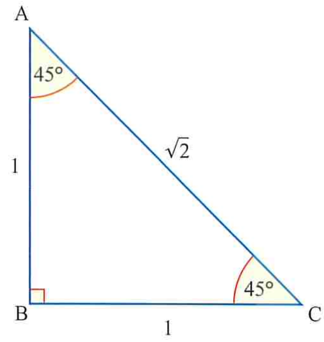


Figure 6.39

Consider an equilateral triangle of side length 2 (Figure 6.40(a)).

By adding an angle bisector we get two congruent triangles (Figure 6.40(b)).

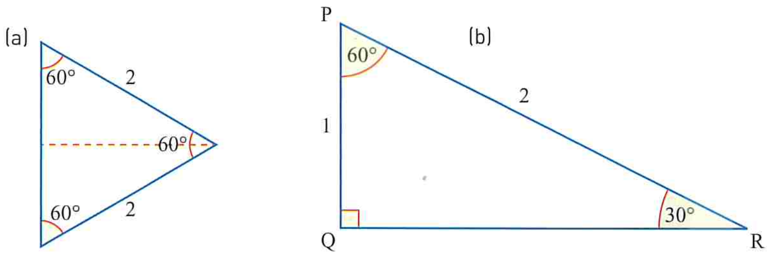


Figure 6.40

Using Pythagoras' theorem

$$QR^2 = 2^2 - 1^2$$
$$QR = \sqrt{3}$$

Using the trig ratios this gives us

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \sqrt{3}$$

Example 6.8

Do not use a calculator for this question.

Work out the exact value of y.

Give your answer in the form $p + q\sqrt{3}$ where p and q are integers.

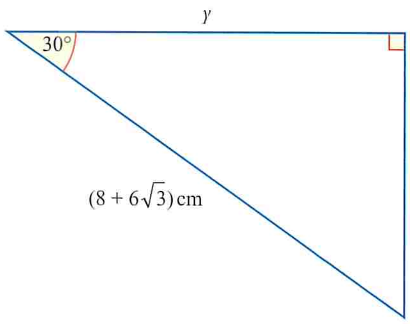


Figure 6.41

Solution

$$\cos 30° = \frac{y}{8+6\sqrt{3}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{y}{8+6\sqrt{3}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} \times (8+6\sqrt{3}) = y$$

$$\Rightarrow \quad 4\sqrt{3}+9 = y$$

$$y = 9+4\sqrt{3}$$

Exercise 6D

Use of a calculator is not allowed.

① Work out the exact value of x in each of the following.
Give answers in their simplest form.

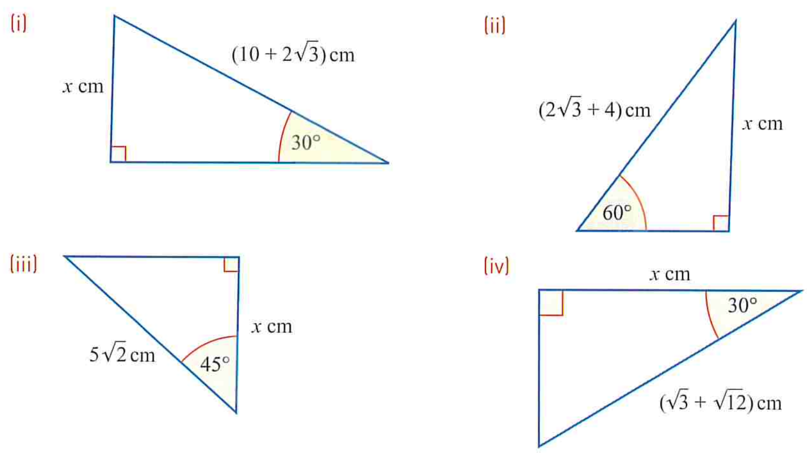


Figure 6.42

② Look at Figure 6.43. Show that y is an integer.

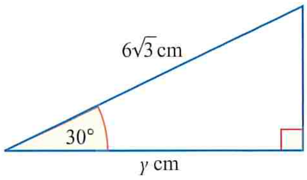


Figure 6.43

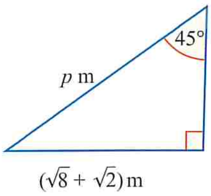


Figure 6.44

③ Look at Figure 6.44. Show that p is an integer.

④ Look at Figure 6.45.

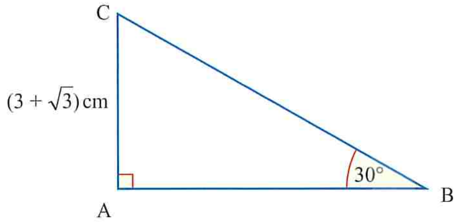


Figure 6.45

Work out the area of triangle ABC.

Give your answer in the form $p + q\sqrt{3}$ where p and q are integers.

⑤ Look at Figure 6.46. Work out the exact value of CD.

Give your answer in the form $k\sqrt{6}$ where k is an integer.

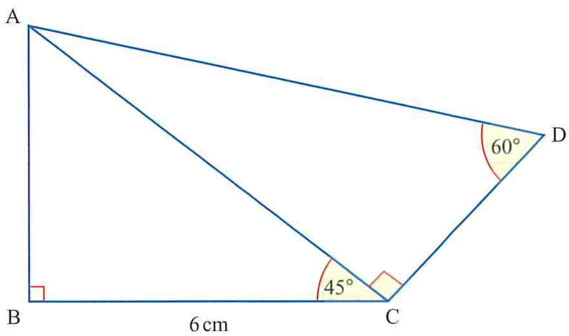


Figure 6.46

RWC ⑥ A ski-lift is spanning a valley in the Alps, rising from a height of 2039 m to a height of 2364 m over a horizontal distance of 325 m. What is the angle of elevation of the ski lift?

RWC ⑦ The centrepiece of a show garden has been designed as a square of side 3 metres surrounded by four equilateral triangles as shown in Figure 6.47.

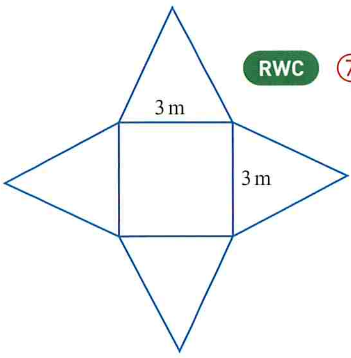


Figure 6.47

(i) The centre square is to be planted with small shrubs which each require a square area of side 30 cm. How many shrubs are required?

(ii) The triangular areas are to be planted with bedding plants, each requiring an area of approximately 100 cm². Approximately how many bedding plants will be required?

(iii) The bedding plants are sold in boxes of 12 and the head gardener decides to order 5% extra plants to allow for ones which might not be up to standard. How many boxes does he need to order?

RWC ⑧ Figure 6.48 shows a vertical building standing on horizontal ground. The points A, B and C are in a straight line on horizontal ground and AC = 30 m. The point T is at the top of the building and CT is vertical. The angles of elevation of T from A and B are 30° and 60° respectively.

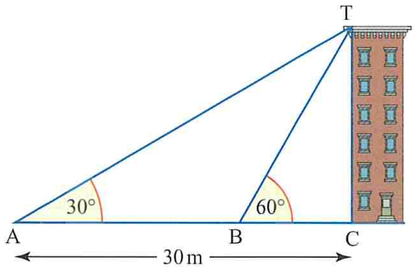


Figure 6.48

(i) Calculate the exact value of the height CT of the building.

(ii) Work out the distances BC and AB.

7 Trigonometric functions for angles of any size

By convention, angles are measured anticlockwise from the positive x-axis (Figure 6.49). Anticlockwise is taken to be positive and clockwise to be negative.

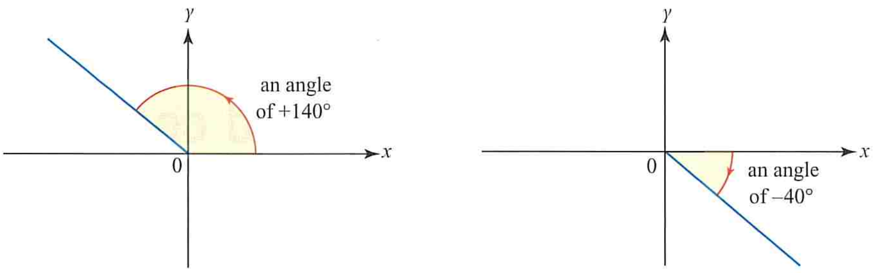


Figure 6.49

The only exception is for compass bearings, which are measured clockwise from the north.

Definitions of the trigonometric functions, sin, cos and tan

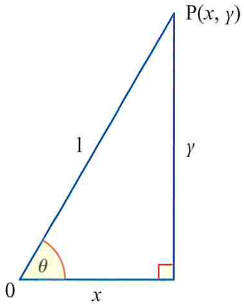


Figure 6.50

First look at the right-angled triangle in Figure 6.50 which has a hypotenuse of unit length.

Sin, cos and tan are defined as the following ratios.

$$\sin\theta = \frac{y}{1} = y \qquad \cos\theta = \frac{x}{1} = x \qquad \tan\theta = \frac{y}{x}$$

We can extend these definitions to angles beyond 90°.

Imagine the angle θ situated at the origin, as in Figure 6.51, and allow θ to take any value. The vertex marked P has coordinates $(\cos\theta, \sin\theta)$ and can now be anywhere on the unit circle.

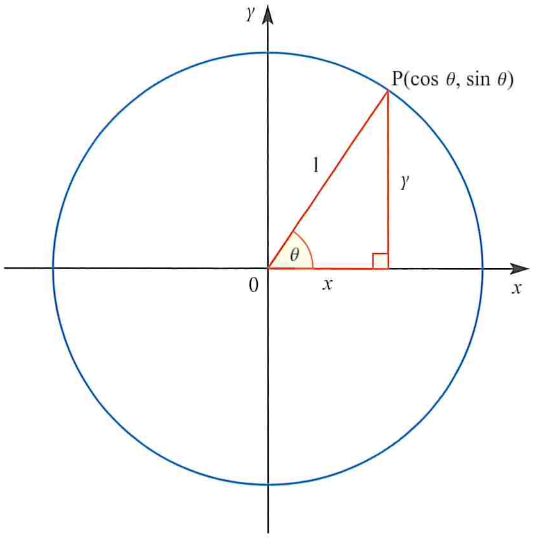


Figure 6.51

You can now see that these definitions can be applied to *any* angle θ, whether it is positive or negative, and whether it is less than or greater than 90°.

$$\sin\theta = y \quad \cos\theta = x \quad \tan\theta = \frac{y}{x}$$

For some angles, x or y (or both) will take a negative value, so the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ will vary accordingly.

8 The sine and cosine graphs

Look at Figure 6.52. There is a unit circle and angles have been drawn at intervals of 30°. The resulting y-coordinates are plotted relative to the axes on the right. They have been joined with a continuous curve to give the graph of $\sin\theta$ for $0 \leqslant \theta \leqslant 360°$.

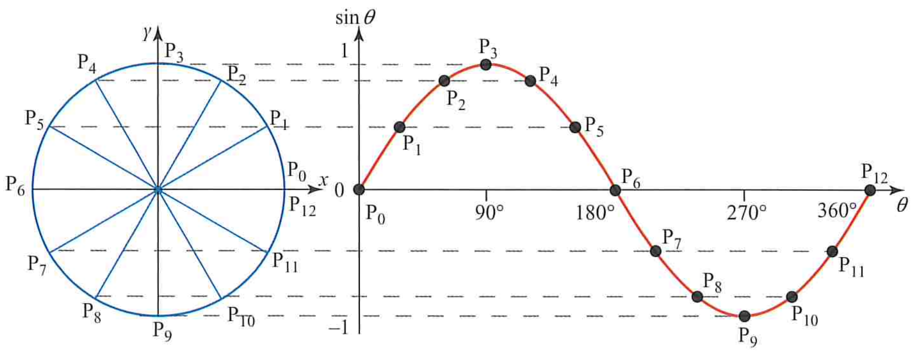


Figure 6.52

Continuing this process for angles 390°, 420°, … and angles −30°, −60°, … you get the graph of $y = \sin\theta$.

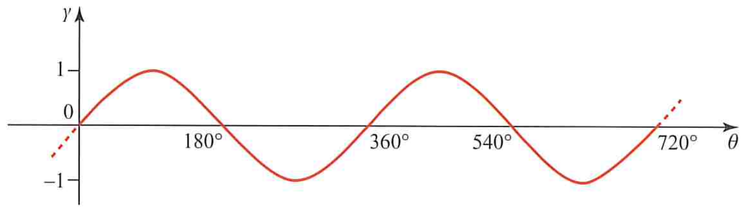


Figure 6.53

Since the curve repeats itself every 360°, as shown in Figure 6.53, the sine function is described as *periodic* with *period* 360°.

In a similar way you can transfer the x-coordinates onto a set of axes to obtain the cosine graph. This is most easily illustrated if you first rotate the circle through 90° anticlockwise.

Figure 6.54 shows this new orientation, together with the resulting graph.

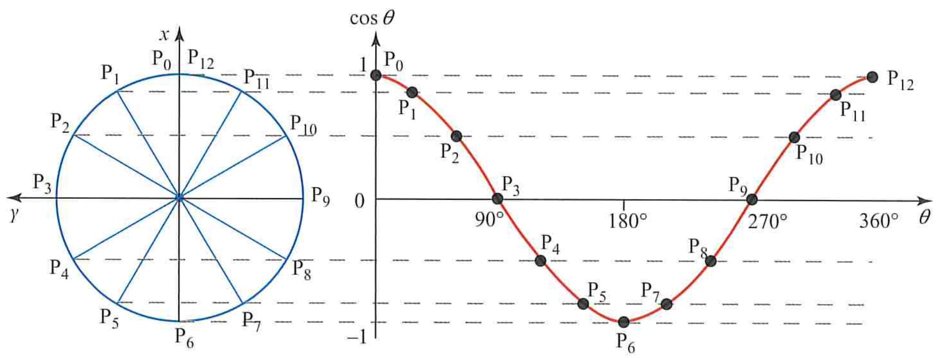


Figure 6.54

For angles beyond this interval the cosine graph repeats itself periodically, with a period of 360°.

Notice that the graphs of $\sin\theta$ and $\cos\theta$ have exactly the same shape. The cosine graph can be obtained by translating the sine graph 90° to the left, as shown in Figure 6.55.

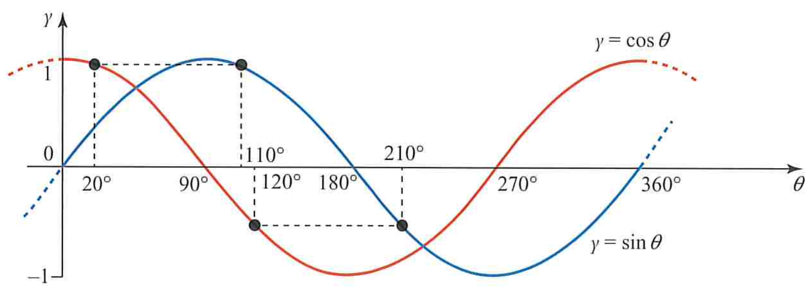


Figure 6.55

9 The tangent graph

The value of $\tan\theta$ can be worked out from the definition $\tan\theta = \frac{y}{x}$ or by using $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Discussion points

- The function $\tan\theta$ is undefined for $\theta = 90°$. What does *undefined* mean?
- How can you tell that $\tan 90°$ is undefined?
- For which **other** values of θ is $\tan\theta$ undefined?

The graph of $\tan\theta$ is shown in Figure 6.56. The dotted lines $\theta = \pm 90°$ and $\theta = 270°$ are *asymptotes*; they are not actually part of the curve.

Discussion point

- How would you describe an asymptote to a friend?

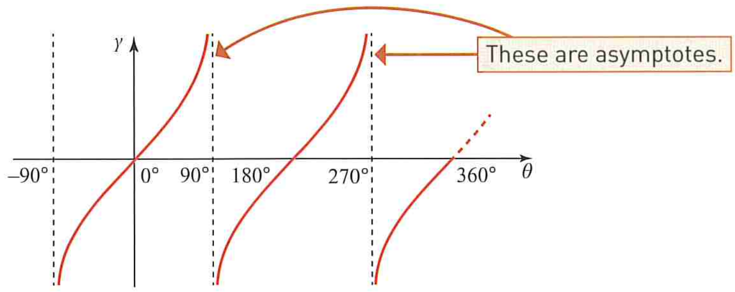


Figure 6.56

Note

It is important to learn the graphs of $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$.

Discussion points

- The graph of $\tan\theta$ is periodic, like those for $\sin\theta$ and $\cos\theta$. What is the period of this graph?
- Show how the part of the curve for $0° \leqslant \theta \leqslant 90°$ can be used to generate the rest of the curve using rotations and translations.

10 Solution of trigonometric equations

Suppose that you want to solve the equation

$$\sin\theta = 0.5$$

You start by pressing the calculator keys

shift | sin | 0 | . | 5 | =

and the answer comes up as 30

Note

The $\sin^{-1}$ key may also be labelled invsin or arcsin.

If your calculator does not give the answer 30 then it might be in the wrong angle setting. Check for a D (or DEG) at the top of the screen. If not, then select DEG whilst in SETUP mode.

However, look at the graph of $y = \sin\theta$ (Figure 6.57). You can see that there are other roots as well.

Discussion point

➜ How many roots does the equation have?

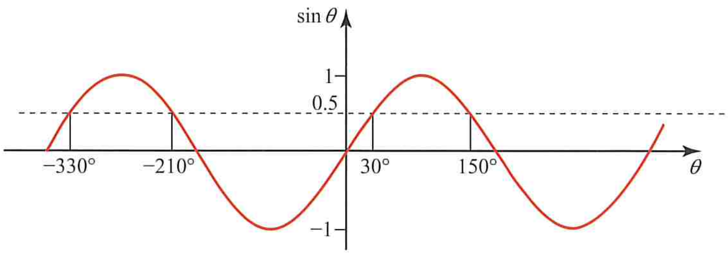


Figure 6.57

The root 30° is called the *principal value.*

Other roots can be found by looking at the graph. The roots for $\sin\theta = 0.5$ are seen to be:

$$\theta = \ldots, -330°, -210°, 30°, 150°, \ldots$$

As the graph is periodic, then the roots repeat every 360°.

Note

A calculator always gives the principal value of the solution. These values are in the range

$0° \leqslant \theta \leqslant 180°$ (cos) $-90° \leqslant \theta \leqslant 90°$ (sin) $-90° < \theta < 90°$ (tan)

Example 6.9

Work out the values of θ in the interval $0° \leqslant \theta \leqslant 360°$ for which $\cos\theta = 0.4$

Solution

$\cos\theta = 0.4 \quad \Rightarrow \quad \theta = 66.4°$ (principal value)

Figure 6.58 shows the graph of $y = \cos\theta$.

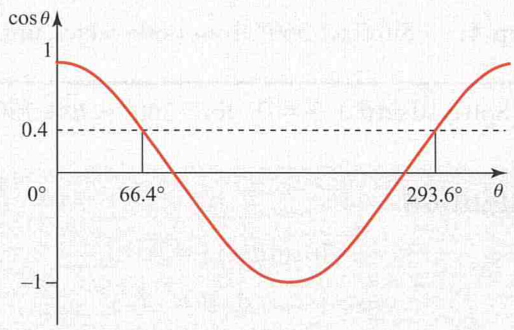


Figure 6.58

The values of θ for which $\cos\theta = 0.4$ are 66.4°, 293.6°

Discussion points

➜ How do we get 293.6° from 66.4°?

➜ Is there a general rule for finding a second angle between $0° \leqslant \theta \leqslant 360°$?

Example 6.10

Work out the values of x in the interval $-360° \leqslant x \leqslant 360°$ for which $6 + 2\tan x = 0$

Solution

$$6 + 2\tan x = 0$$

$$\Rightarrow \quad 2\tan x = -6$$

$$\Rightarrow \quad \tan x = -3$$

$$\Rightarrow \quad x = -71.6° \quad \text{(principal value)}$$

Figure 6.59 shows the graph of $y = \tan x$.

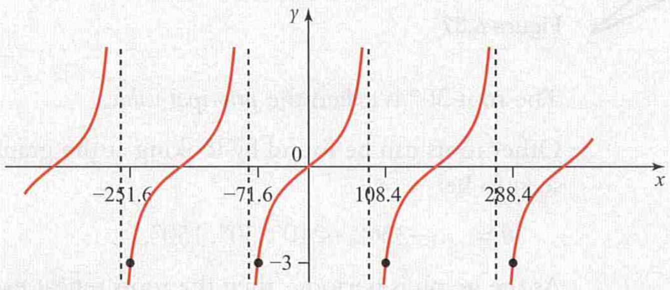


Figure 6.59

The values of x for which $\tan x = -3$ are

$-251.6°, -71.6°, 108.4°, 288.4°$

Short method for solving trigonometric equations for any angle:

Step 1: Use $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ to find the principal value, θ.

Step 2: Work out a second angle using one of the following

$\sin^{-1}$:	$180° - \theta$
$\cos^{-1}$:	$360° - \theta$
$\tan^{-1}$:	$\theta + 180°$

Step 3: Add 360° to both values, until the upper limit is reached.

Step 4: Subtract 360° from both values, until the lower limit is reached.

Example 6.11

Solve $10\sin\theta + 3 = 0$ for $-360° \leqslant \theta \leqslant 720°$.

Solution

$$10\sin\theta + 3 = 0$$

$$\Rightarrow \quad \sin\theta = -0.3$$

Step 1: Principal value $= \sin^{-1}(-0.3)$

$= -17.5°$

Step 2: The second angle is $180° + 17.5° = 197.5°$

Step 3: $-17.5° + 360° = 342.5°$ and $342.5° + 360° = 702.5°$

$197.5° + 360° = 557.5°$

Step 4: $-17.5° - 360° = -377.5°$ (too low)

$197.5° - 360° = -162.5°$

$\therefore \quad \theta = -162.5°, -17.5°, 197.5°, 342.5°, 557.5°$ or $702.5°$

Make sure the graph plotter is set to 'degrees'.

ACTIVITY 6.3

- Use a graph plotter to plot the graph of $y = \sin(x + 10)$.
- How does the graph of $y = \sin(x + 10)$ compare with the graph of $y = \sin x$?
- Then do the same with the graph of $y = \sin(x + 20)$.
- Is it possible to find a graph of the form $y = \sin(x + c)$ which is the same as the graph of $y = \cos x$?
- Write down an identity in the form $\cos x \equiv \sin(x + c)$ where c is a number to be found.
- Go through the same process with the cosine graph to find a similar identity of the form $\sin x \equiv \cos(x + d)$.

Note

Knowledge of these graphs and the corresponding identities will not be examined in this specification. However, students who go on to study Mathematics at A-Level will meet them again.

Exercise 6E

Give answers to 1 decimal place where necessary.

① Solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(i) $\cos\theta = 0.5$ (ii) $\tan\theta = 1$ (iii) $\sin\theta = \frac{\sqrt{3}}{2}$

(iv) $\sin\theta = -0.5$ (v) $\cos\theta = 0$ (vi) $\tan\theta = -5$

(vii) $\tan\theta = 0$ (viii) $\cos\theta = -0.54$ (ix) $\sin\theta = 1$

② Solve the following equations for $-180° \leqslant \theta \leqslant 180°$.

(i) $3\cos\theta = 2$ (ii) $7\sin\theta = 5$ (iii) $3\tan\theta = 8$

(iv) $6\sin\theta + 5 = 0$ (v) $5\cos\theta + 2 = 0$ (vi) $5 - 9\tan\theta = 10$

PS ③ Solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(i) $\sin^2\theta = 0.75$ (ii) $\cos^2\theta = 0.5$ (iii) $\tan^2\theta = 1$

PS ④ (i) Factorise $2x^2 + x - 1$

(ii) Hence solve $2x^2 + x - 1 = 0$

(iii) Use your results to solve these equations for $-360° \leqslant \theta \leqslant 360°$.

(a) $2\sin^2\theta + \sin\theta - 1 = 0$

(b) $2\cos^2\theta + \cos\theta - 1 = 0$

(c) $2\tan^2\theta + \tan\theta - 1 = 0$

$\sin^2\theta$ is alternative notation for $(\sin\theta)^2$

PS ⑤ Solve the following equations for $-180° \leqslant x \leqslant 180°$.

(i) $\tan^2 x - 3\tan x = 0$ (ii) $1 - 2\sin^2 x = 0$

(iii) $3\cos^2 x + 2\cos x - 1 = 0$ (iv) $2\sin^2 x = \sin x + 1$

PS ⑥ *Do not use a calculator in this question.*

Solve the following equations for $-360° < x < 360°$.

(i) $\tan x = \sqrt{3}$

(ii) $2\sin x = 1$

(iii) $\sqrt{2}\cos x - 1 = 0$

(iv) $2\sin x = \sqrt{3}$

(v) $\tan^2 x - \tan x = 0$

(vi) $4\cos x = \sqrt{12}$

PS ⑦ Solve $(\cos\theta - 1)(\cos\theta + 2)(2\cos\theta - 1) = 0$ for $0° \leqslant \theta \leqslant 360°$.

PS ⑧ (i) Given that $f(x) = 2x^3 - x^2 - 3x - 1$, calculate $f\left(-\frac{1}{2}\right)$.

(ii) Hence solve $2\sin^3\theta - \sin^2\theta - 3\sin\theta - 1 = 0$ for $-180° \leqslant \theta \leqslant 180°$.

11 Trigonometric identities

Remember the earlier definitions for trigonometric functions of angles of any magnitude

$$\sin\theta = y \quad \cos\theta = x \quad \tan\theta = \frac{y}{x}$$

where the angle θ was defined by a point $P(x, y)$ on a circle of unit radius (Figure 6.60).

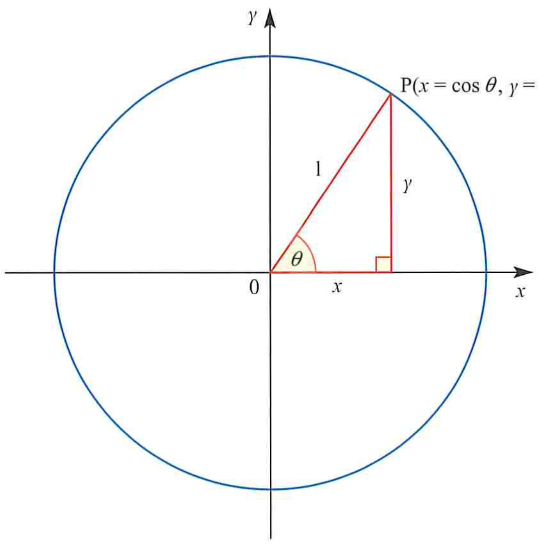


Figure 6.60

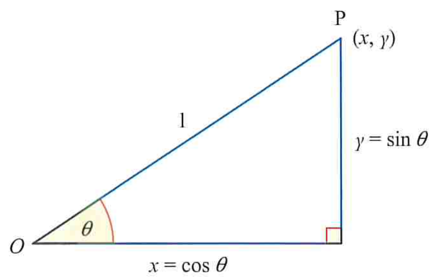


Figure 6.61

Applying Pythagoras' theorem to the triangle it can be seen that

$$\sin^2\theta + \cos^2\theta = 1$$

If θ is not acute, then x and/or y would be negative, but squaring produces the same result.

So the rule is correct for any value of θ.

As is the rule $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ which was seen earlier in this chapter.

Example 6.12

Solve the equation $2\cos^2\theta - \sin\theta - 1 = 0$ for values of θ in the range 0° to 360°.

Solution

$$
\begin{aligned}
& 2\cos^2\theta - \sin\theta - 1 = 0 \\
\Rightarrow\ & 2(1 - \sin^2\theta) - \sin\theta - 1 = 0 \\
\Rightarrow\ & 2 - 2\sin^2\theta - \sin\theta - 1 = 0 \\
\Rightarrow\ & 0 = 2\sin^2\theta + \sin\theta - 1 \\
\Rightarrow\ & 0 = (2\sin\theta - 1)(\sin\theta + 1) \\
\Rightarrow\ & \sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -1 \\
\Rightarrow\ & \theta = 30°,\ 150° \quad \text{or} \quad \theta = -90°,\ 270° \\
\therefore\ & \theta = 30°,\ 150° \quad \text{or} \quad 270°
\end{aligned}
$$

$\sin^2\theta + \cos^2\theta = 1$ can be rearranged into $\cos^2\theta = 1 - \sin^2\theta$.
So, here the $\cos^2\theta$ has been replaced with $1 - \sin^2\theta$, leaving an equation in $\sin\theta$.

Example 6.13

Show that $\dfrac{\tan x \cos x}{\sqrt{1 - \cos^2 x}}$ simplifies to 1

Solution

$$
\begin{aligned}
\frac{\tan x \cos x}{\sqrt{1 - \cos^2 x}} &= \frac{\frac{\sin x}{\cos x}\cos x}{\sqrt{\sin^2 x}} \\
&= \frac{\sin x}{\sin x} \\
&= 1
\end{aligned}
$$

Example 6.14

(i) Prove that $\cos^2 x - \sin^2 x \equiv 2\cos^2 x - 1$

(ii) Hence, solve $\cos^2 x - \sin^2 x = 0.5$ for $0° \leqslant x \leqslant 360°$.

Solution

(i) Start with one side of the identity and, step-by-step, change it into the other side.

$$
\begin{aligned}
\cos^2 x - \sin^2 x &\equiv \cos^2 x - (1 - \cos^2 x) \\
&\equiv \cos^2 x - 1 + \cos^2 x \\
&\equiv 2\cos^2 x - 1
\end{aligned}
$$

The equivalence (or identity) sign is used to indicate that the equation is true for all values of x.

(ii) Using part (i), the equation can be rewritten as

$$2\cos^2 x - 1 = 0.5$$
$$\Rightarrow \quad 2\cos^2 x = 1.5$$
$$\Rightarrow \quad \cos^2 x = \frac{3}{4}$$
$$\Rightarrow \quad \cos x = \pm\sqrt{\frac{3}{4}}$$
$$\Rightarrow \quad \cos x = \frac{\sqrt{3}}{2} \quad \text{or} \quad \cos x = -\frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad x = 30°, 330° \quad \text{or} \quad x = 150°, 210°$$

Summary

$\sin^2\theta + \cos^2\theta \equiv 1$ and $\sin^2\theta \equiv 1 - \cos^2\theta$ and $\cos^2\theta \equiv 1 - \sin^2\theta$

$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin\theta \equiv \cos\theta\tan\theta$ and $\cos\theta \equiv \dfrac{\sin\theta}{\tan\theta}$

Exercise 6F

① For each of the equations (i)–(v):

(a) use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to rewrite the equation in a form involving only one trigonometric function

(b) factorise, and hence solve, the resulting equation for $0° \leqslant \theta \leqslant 360°$.

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

(ii) $\sin^2\theta + \cos\theta + 1 = 0$

(iii) $2\sin^2\theta - \cos\theta - 1 = 0$

(iv) $\cos^2\theta + \sin\theta = 1$

(v) $1 + \sin\theta - 2\cos^2\theta = 0$

② For each of the equations (i)–(iii):

(a) use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to rewrite the equation in a form involving only one trigonometric function

(b) use the quadratic formula to solve the resulting equation for $0° \leqslant \theta \leqslant 180°$.

(i) $\sin^2\theta - 2\cos\theta + 1 = 0$

(ii) $\cos^2\theta - \sin\theta = 0$

(iii) $\sin^2\theta - 3\cos\theta = 0$

③ (i) Use the identity

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

to rewrite the equation $\sin\theta = 2\cos\theta$ in terms of $\tan\theta$.

(ii) Hence solve the equation $\sin\theta = 2\cos\theta$ for $0° \leqslant \theta \leqslant 180°$.

④ Use the identity

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

to solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(i) $2\sin\theta + \cos\theta = 0$

(ii) $\sqrt{3}\tan\theta = 2\sin\theta$

(ii) $4\cos\theta\tan\theta = 1$

PS ⑤ Write the following in terms of $\sin x$.

(i) $\cos^2 x \tan^2 x$ (ii) $\tan x \cos^3 x$ (iii) $\cos x\,(2\cos x - 3\tan x)$

PS ⑥ Show that $(3\sin x)(\sin x + 2) - 3(2\sin x - \cos^2 x)$ simplifies to an integer.

PS ⑦ Prove the following identities.

(i) $\tan x\sqrt{1 - \sin^2 x} \equiv \sin x$

(ii) $\dfrac{1 - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x$

(iii) $(1 + \sin x)(1 - \sin x) \equiv \cos^2 x$

(iv) $\dfrac{2\sin x\cos x}{\tan x} \equiv 2 - 2\sin^2 x$

PS ⑧ Solve $5\sin x\left(\sin x + \cos x\right) = 3$ for $0° < x < 360°$.

Hint: Replace 3 with 3×1 and then replace the 1 with $\sin^2 x + \cos^2 x$.

FUTURE USES

Trigonometric functions are explored to a greater depth in A-Level Mathematics, including the use of various trig identities and the study of inverse trig functions. Trigonometry is used in many areas, including mechanics when resolving vectors such as forces and velocities.

It is also used extensively in A-Level Further Mathematics, to describe complex numbers (a combination of real and imaginary numbers), to describe transformations, and many other applications.

In A-Level Further Mathematics, you will also study the hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$.

REAL-WORLD CONTEXT

Trigonometry has many real-world applications, including every aspect of engineering. It is also essential to architects and surveyors. Space exploration and the motion/positioning of satellites would not be possible without trigonometry. Mobile telephones, video games, and computers in general, make much use of this vital area of mathematics. In fact, ancient civilisations were aware of its usefulness, and made use of it to achieve amazing feats of construction, many of which are still standing to this day.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- use trigonometry in a right-angled triangle
  - to find an angle when you know any two sides
  - to find the other sides or angle when you know the length of one side and an angle
- use Pythagoras' theorem in two dimensions
  - in the form $a^2 + b^2 = c^2$
- apply the following angle facts
  - vertically opposite angles are equal
  - adjacent angles on a straight line add up to 180°
  - alternate angles are equal
  - corresponding angles are equal
  - interior angles add up to 180°
- apply the following circle theorems
  - the angle at the centre is double the angle at the circumference
  - angle in a semi-circle = 90°
  - angles in the same segment are equal
  - opposite angles of a cyclic quadrilateral add up to 180°
  - alternate segment theorem
- construct a formal geometric proof for problems involving triangles and circles
- solve practical problems in two dimensions (e.g. a ladder against a wall)
- use a calculator to find
  - the sin, cos or tan of any angle
  - an angle, given the sin, cos or tan ratio
- sketch and recognise the graphs of sin, cos or tan for any angle
- solve trigonometric equations
- recognise and use the trigonometric identities linking sin, cos and tan.

KEY POINTS

1 In a right-angled triangle Pythagoras' theorem gives

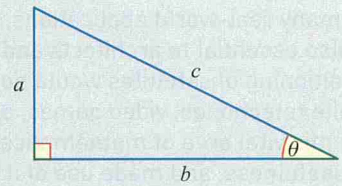


Figure 6.62

$c^2 = a^2 + b^2$

2 Using the triangle above gives the definitions:

$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c} \qquad \tan\theta = \frac{a}{b}$

4 $\sin 45° = \frac{1}{\sqrt{2}}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\tan 45° = 1$

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$

5 In a geometrical proof, show all your working and give unambiguous reasons for each stage.

6 Trig graphs:

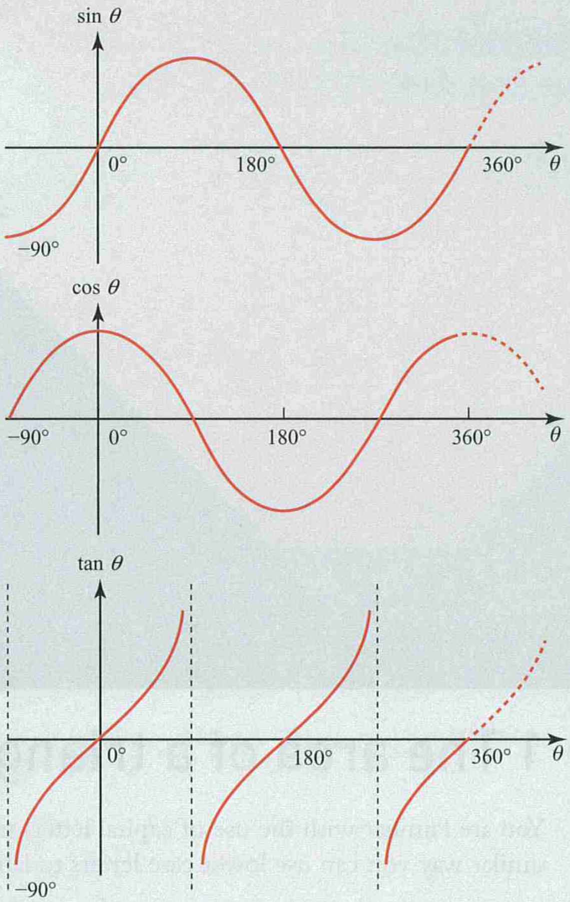


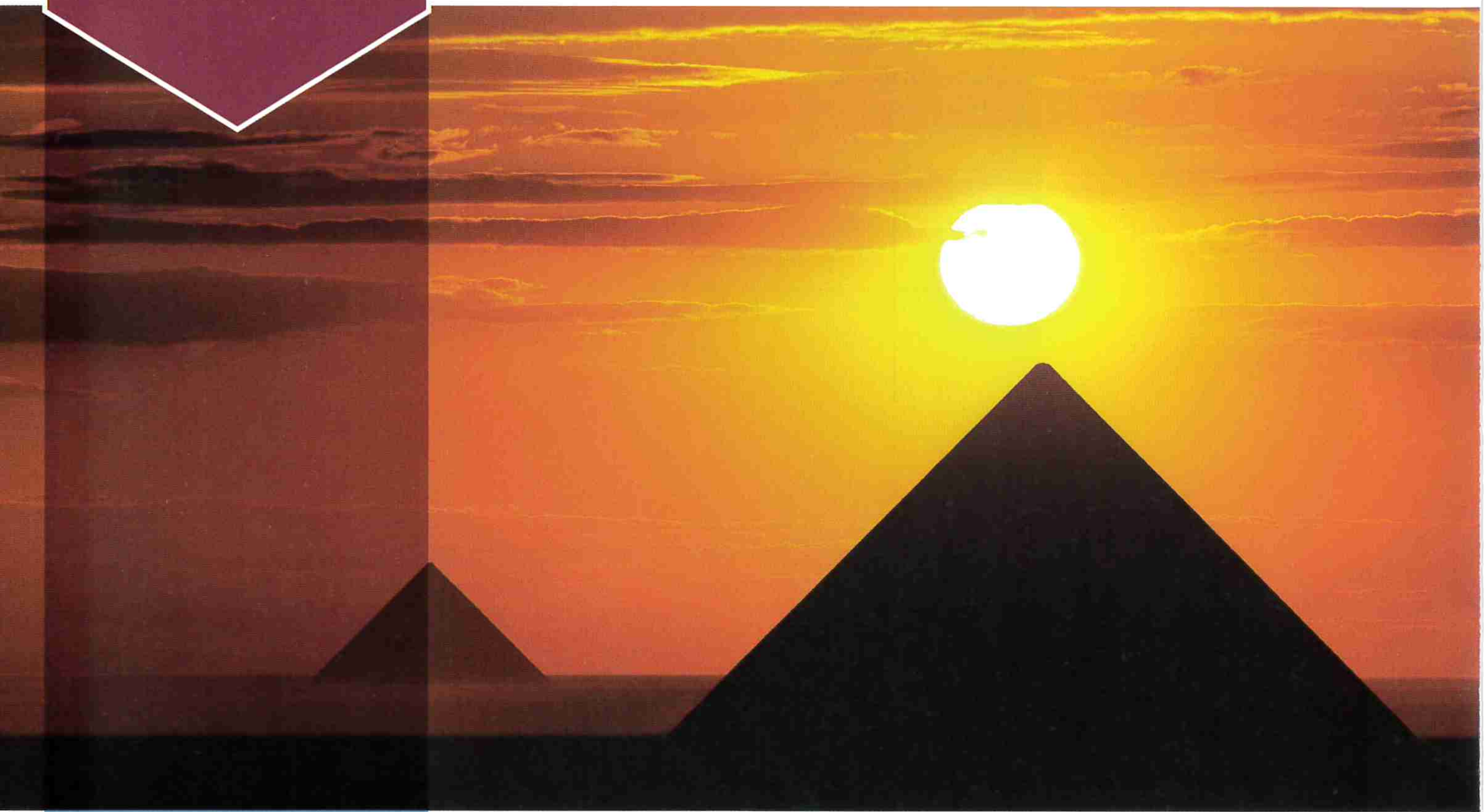
Figure 6.63

7 Trig identities:

$\sin^2\theta + \cos^2\theta \equiv 1$ and $\sin^2\theta \equiv 1 - \cos^2\theta$ and $\cos^2\theta \equiv 1 - \sin^2\theta$

$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ and $\sin\theta \equiv \cos\theta\tan\theta$ and $\cos\theta \equiv \frac{\sin\theta}{\tan\theta}$

7 Geometry II



What we know is a drop; what we don't know is an ocean.

Isaac Newton

1 The area of a triangle

You are familiar with the use of capital letters to label the vertices of a triangle. In a similar way you can use lower case letters to label the sides.

a denotes the length of the side opposite angle A, b is the length of the side opposite angle B, and c is the length of the side opposite angle C.

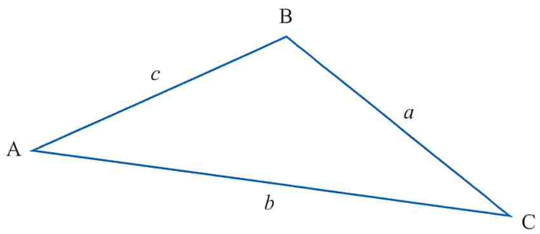


Figure 7.1

Using this notation, for any triangle ABC the area is given by the formula

$$\text{area} = \frac{1}{2}bc\sin A.$$

Proof

Figure 7.2 shows a triangle ABC. The perpendicular CD is the height h corresponding to AB as base.

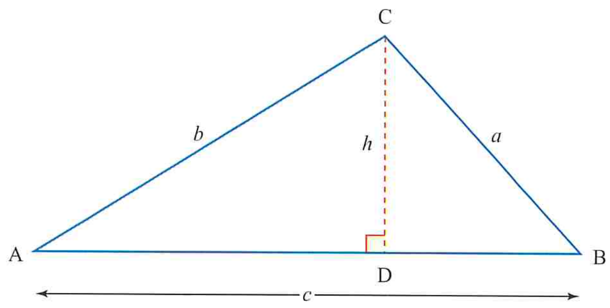


Figure 7.2

Using area of a triangle equals half its base times its height,

$$\text{area} = \frac{1}{2}ch \qquad ①$$

In triangle ACD

$$\sin A = \frac{h}{b}$$

$$\Rightarrow \qquad h = b \sin A$$

Substituting in ① gives

$$\text{area} = \frac{1}{2}bc \sin A$$

Note

Taking the other two points in turn as the top of the triangle gives equivalent results:

$$\text{area} = \frac{1}{2}ca \sin B$$

and

$$\text{area} = \frac{1}{2}ab \sin C.$$

The formula may be easier to remember as half the product of two sides times the sine of the angle between them.

Example 7.1

Figure 7.3 shows a regular pentagon, PQRST, inscribed in a circle, centre C, radius 8 cm. Calculate the area of

(i) triangle CPQ

(ii) the pentagon.

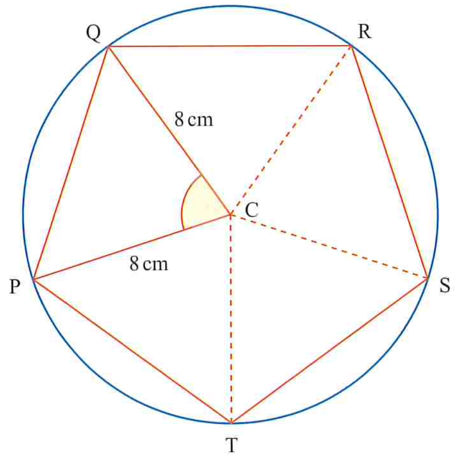


Figure 7.3

Solution

(i) angle PCQ $= 360° \div 5$

$= 72°$

area PCQ $= \frac{1}{2} \times 8 \times 8 \times \sin 72°$

$= 30.4338...$

$= 30.4\,\text{cm}^2$ (1 d.p.)

(ii) area PQRST $= 5 \times 30.4338...$

$= 152.169...$

$= 152.2\,\text{cm}^2$ (1 d.p.)

Example 7.2

Figure 7.4 shows an isosceles triangle with an area of $24\,\text{cm}^2$ and one angle of 40°. Calculate the lengths of the two equal sides.

Solution

Let the equal sides be of length x cm.

Using area $= \frac{1}{2}ab\sin C$

$\therefore \quad 24 = \frac{1}{2} \times x \times x \times \sin 40°$

$\Rightarrow \quad x^2 = \frac{48}{\sin 40°}$

$\Rightarrow \quad x = 8.64\,\text{cm}$ (3 s.f.)

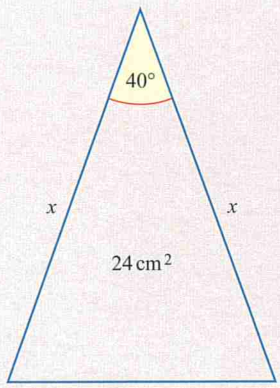


Figure 7.4

Exercise 7A

Where necessary leave answers approximated to 3 significant figures.

① Work out the area of each of the following triangles.

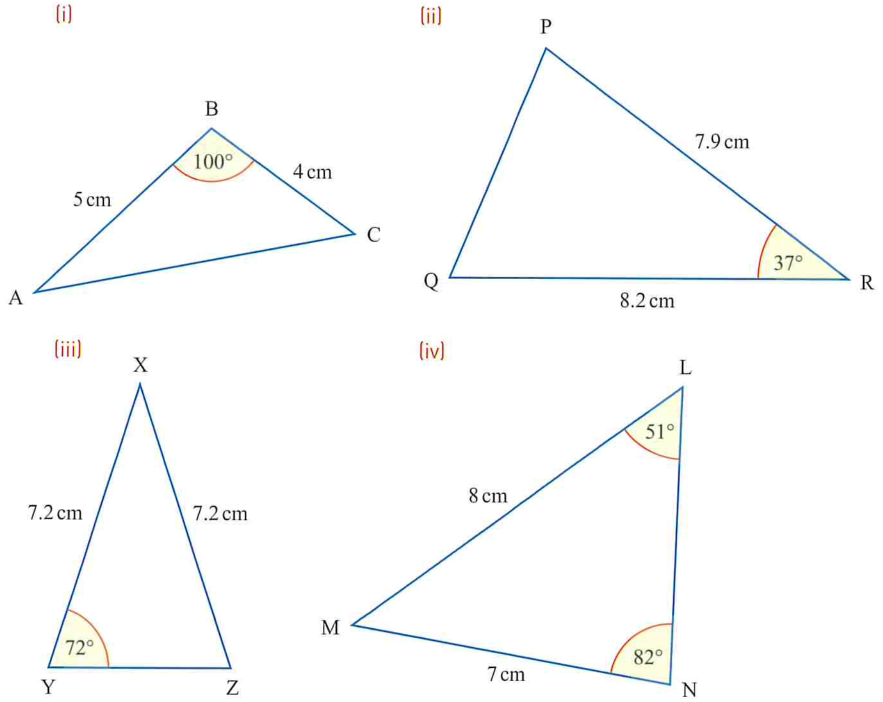


Figure 7.5

PS ② A regular hexagon is made up of six equilateral triangles. Work out the area of a regular hexagon of side 7 cm.

PS ③ A pyramid on a square base has four identical triangular faces which are isosceles triangles with equal sides 9 cm and equal angles 72°.

(i) Work out the area of a triangular face.
(ii) Work out the length of a side of the base.
(iii) Hence work out the total surface area of the pyramid.

PS ④ A tiler wishes to estimate the number of triangular tiles needed to tile an area of $10\,\text{m}^2$. The dimensions of each tile are shown in the diagram.

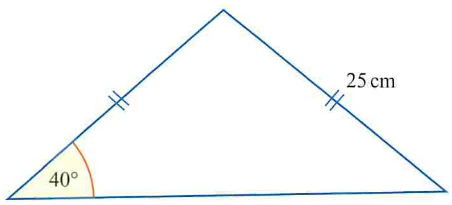


Figure 7.6

(i) Work out the area of a tile.

The tiler then divides $10\,\text{m}^2$ by this area and rounds to the next whole number.

(ii) What result would this give?
(iii) Explain what is wrong with this estimate.

PS ⑤ A regular tetrahedron has four faces, each of which is an equilateral triangle of side 10 cm. Work out the total surface area of the tetrahedron.

PS ⑥ The area of a rhombus is $\sqrt{48}\,\text{cm}^2$. Given also that one of its interior angles is 120°, work out the length of its shortest diagonal.

PS ⑦ A square with sides of length 2 cm has the same area as an equilateral triangle. Work out the side-length of the triangle.

PS ⑧ A circle is drawn inside a square, so that they touch at four points as shown.

A rectangle of dimensions 1 cm × 2 cm is drawn in the corner of the square and touches the circle once. The sides of the rectangle are parallel to the sides of the square. A radius of the circle is drawn to the point where the rectangle meets the circle.

Work out the size of the angle marked θ.

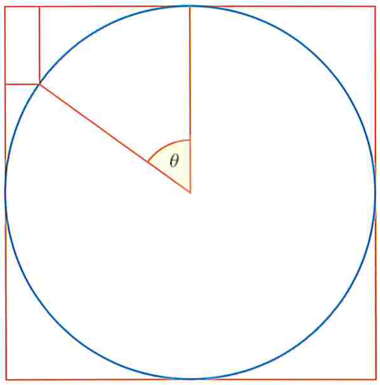


Figure 7.7

The following two trigonometric rules can be used in any triangle, which makes them particularly useful when dealing with scalene triangles.

2 The sine rule

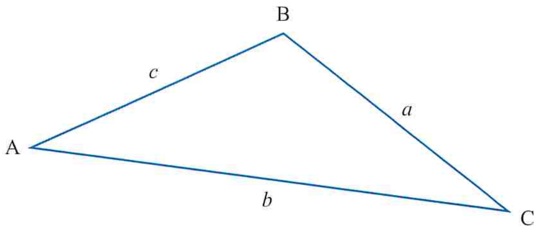


Figure 7.8

You have already seen that for any triangle ABC

$$\text{area} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$\Rightarrow \frac{bc\sin A}{abc} = \frac{ca\sin B}{abc} = \frac{ab\sin C}{abc}$$

$$\Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This is one form of the *sine rule* and is the version that is easier to use if you want to work out the size of an angle.

Inverting this gives

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

which is better when you need to work out the length of a side.

Discussion point

➜ Why is the inverted form of the sine rule better when you want to work out the length of a side?

Note

If a triangle is right-angled then it is much simpler to use the basic trig ratios and/or Pythagoras' theorem. However, the sine and cosine rules are still applicable.

Example 7.3

Work out the length of the side BC in the triangle shown in Figure 7.9.

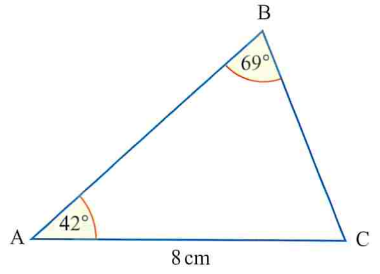


Figure 7.9

> **!** When using the sine rule to work out the size of an angle, you need to be careful because sometimes there are two possible answers, as in Example 7.4. The reason this problem occurs is that for any positive sine ratio there are two possible angles in the range 0° to 180°, except sin 90° = 1.

Solution

Using the sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{a}{\sin 42°} = \frac{8}{\sin 69°}$$

$$\Rightarrow \quad a = \frac{8 \sin 42°}{\sin 69°}$$

$$= 5.733\,887\ldots$$

$$\therefore \quad \text{side BC} = 5.7\,\text{cm (1 d. p.)}$$

> It is advisable to do the calculation entirely on your calculator, and round only the final answer.

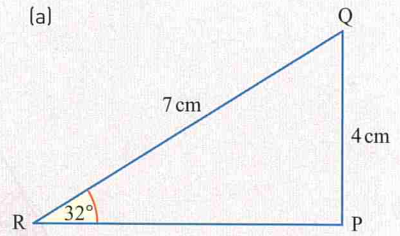
Example 7.4

Work out the size of the angle P in the triangle PQR, given that R = 32°, $r = 4$ cm and $p = 7$ cm where r and p are the lengths of the sides opposite angles R and P respectively.

> **!** Always check that the second option is a valid solution. Sometimes one of the options gives an impossible triangle. For example, if a solution has an angle sum which is greater than 180° then we can reject it. Also, the longest side must be opposite the largest angle, and the shortest side must be opposite the smallest angle.

Solution

The sine rule for ΔPQR is



$$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$$

$$\therefore \quad \frac{\sin P}{7} = \frac{\sin 32°}{4}$$

$$\Rightarrow \quad \sin P = 0.927\,358\,712$$

$$\Rightarrow \quad P = 68.0° \text{ (1 d.p.) or } P = 180° - 68.0° = 112.0° \text{ (1 d.p.)}$$

Both solutions are possible as indicated in Figure 7.10(b).

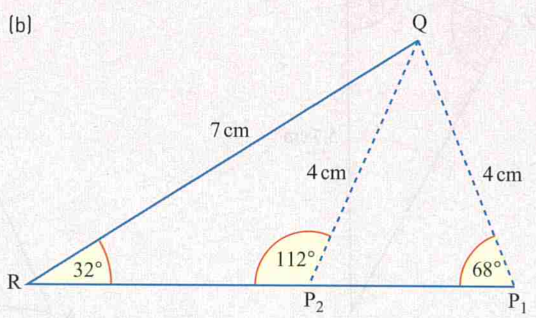


Figure 7.10

Note

Students may have met this situation when studying congruent triangles in GCSE maths.

If two triangles have SSS, SAS, ASA (or AAS) or RHS in common then they are congruent. However, if they have ASS in common (as in the above example), then they are not necessarily congruent, as there are two possible triangles.

ACTIVITY 7.1

Figure 7.11 shows triangle XYZ with $XY = 6\,\text{cm}$, $XZ = 8\,\text{cm}$ and $\angle XYZ = 78°$. What happens when you use the sine rule to calculate the remaining angles?

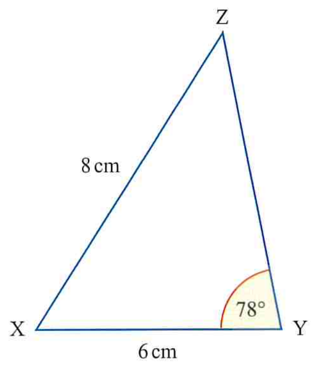


Figure 7.11

Prior knowledge

Many trigonometric problems involve the use of bearings, which is covered in the GCSE specification.

Exercise 7B

Where necessary leave answers approximated to 3 significant figures.

① Work out the length x in each of these triangles.

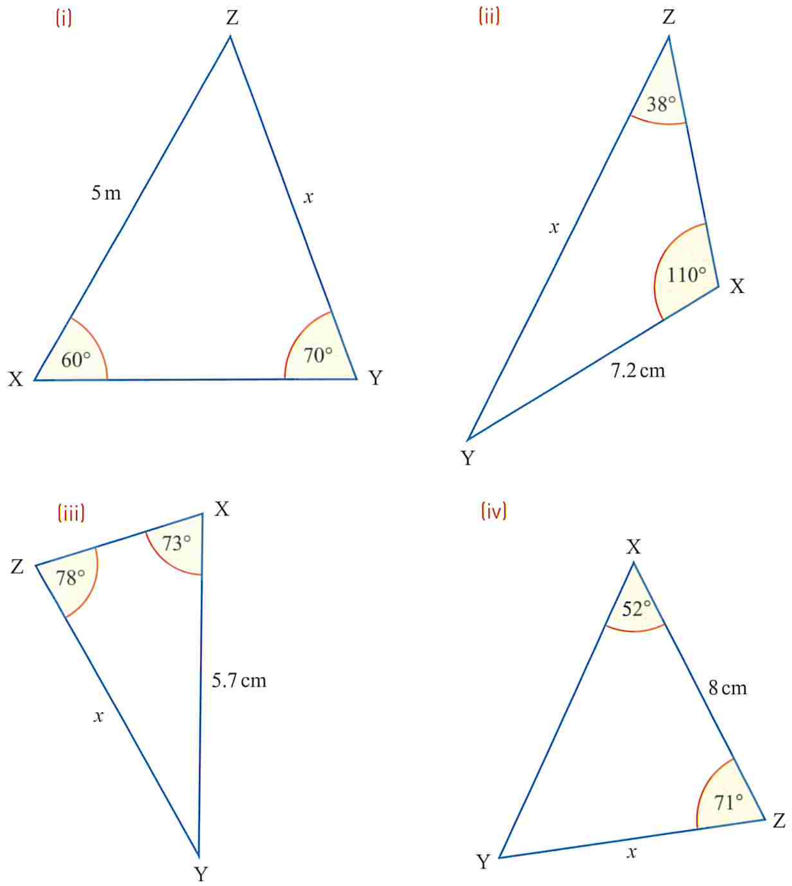


Figure 7.12

② Work out the size of the angle θ in each of these triangles.

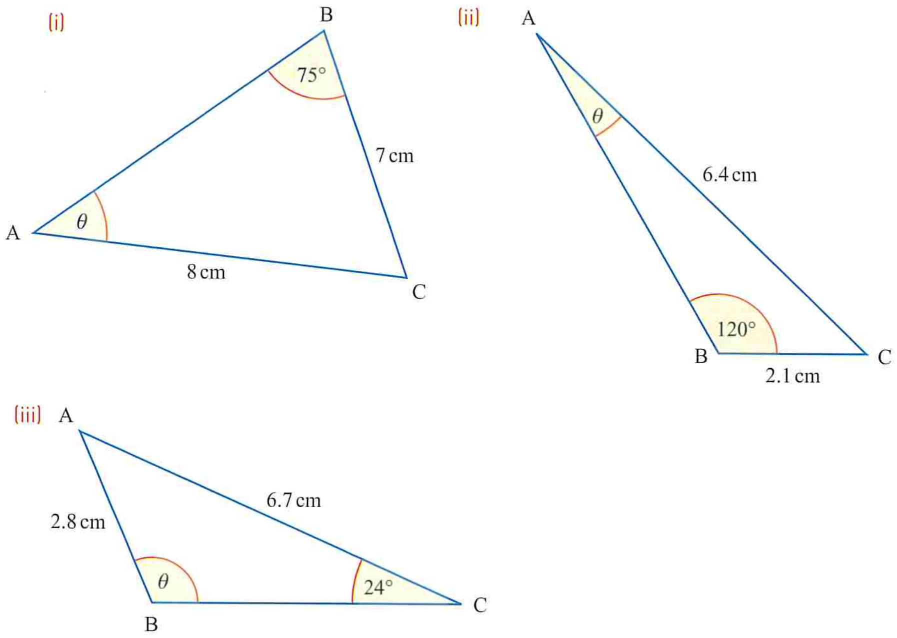


Figure 7.13

③ Work out the size of the angle marked x in the quadrilateral shown.

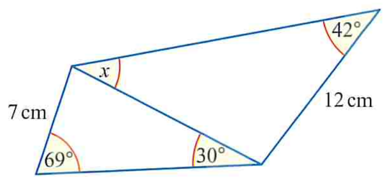


Figure 7.14

④ Tracey walks on a bearing of 132° for 4 km.

She then changes direction and walks on a bearing of 017° until she is due east of her starting position.

How far is she from her starting position?

⑤ The angles of a kite are 122°, 102°, 102° and 34°.

The diagonal which lies along the kite's line of symmetry is 12 cm in length.

Work out the lengths of each of the kite's four sides.

⑥ Anna and Julia are at point P.

Point Q is due North of point P.

They disagree about the shortest route from P to Q.

Anna walks on a bearing of 330°, and then changes to a bearing of 040°, which takes her straight to Q.

Julia walks 3 km on a bearing of 020°, after which she walks on a bearing of 300°, which then takes her straight to Q.

Who took the shorter route?

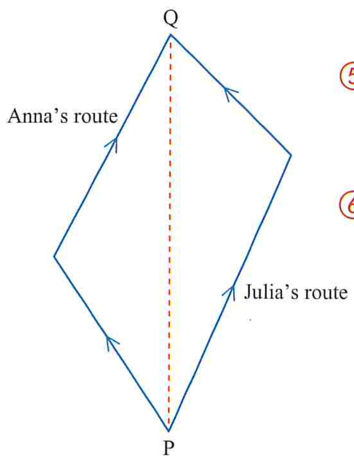


Figure 7.15

3 The cosine rule

Sometimes it is not helpful to use the sine rule with the information you have about a triangle, for example, if you know all three side lengths but none of the angles.

Like the sine rule, the cosine rule can be applied to any triangle, and again there are equivalent versions.

Use this version to work out a side length.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Use this version to work out the size of an angle.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Proof

For the triangle ABC, line CD is perpendicular to side AB as shown in Figure 7.16.

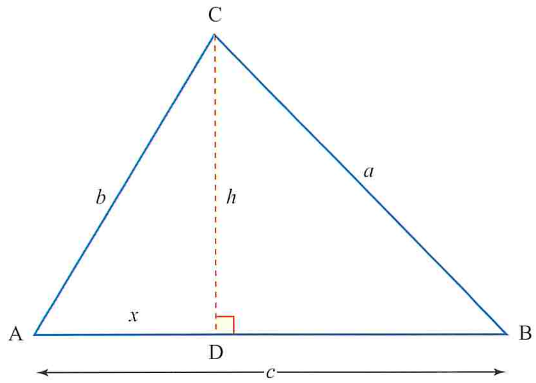


Figure 7.16

This is shorthand notation for 'triangle'.

Pythagoras' theorem.

Pythagoras' theorem.

In Δ ACD

$b^2 = x^2 + h^2$ ①

and $\cos A = \frac{x}{b}$ so $x = b\cos A$ ②

In Δ BCD

$a^2 = (c - x)^2 + h^2$

$\Rightarrow a^2 = c^2 - 2cx + x^2 + h^2$

$\Rightarrow a^2 = c^2 - 2cx + b^2$ using ①

$\Rightarrow a^2 = c^2 - 2cb\cos A + b^2$ using ②

$\Rightarrow a^2 = b^2 + c^2 - 2bc\cos A$ (as required)

Rearranging this gives

$$2bc\cos A = b^2 + c^2 - a^2$$

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

which is the second form of the cosine rule.

Note

Starting with a perpendicular from a different vertex would give the following similar results.

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \text{and} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{and} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 7.5

Work out the length of the side AB in the triangle shown in Figure 7.17.

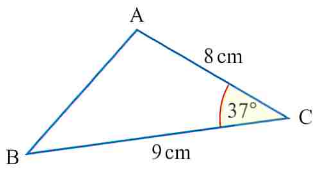


Figure 7.17

Solution

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 9^2 + 8^2 - 2 \times 9 \times 8 \times \cos 37$$

$$= 29.996$$

$$AB = 5.48\,\text{cm} \quad (3\text{ s.f.})$$

There are two common errors when using this formula.

- In a non-calculator paper, evaluate the three terms a^2, b^2 and $2ab\cos C$ separately. A common error is to calculate $a^2 + b^2 - 2ab$ and then multiply by $\cos C$. However, these questions are usually in calculator papers, in which case the whole calculation can be typed into a scientific calculator – this will deal with the priority of operations correctly.
- Another common error is to forget to square root after calculating $a^2 + b^2 - 2ab\cos C$.

Example 7.6

Work out the size of the angle P in the triangle shown in Figure 7.18.

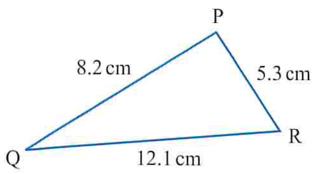


Figure 7.18

Solution

The cosine rule for this triangle can be written as

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

$$\cos P = \frac{5.3^2 + 8.2^2 - 12.1^2}{2 \times 5.3 \times 8.2}$$

$$\cos P = -0.588$$

$$P = 126.0° \quad (1\text{ d.p.})$$

Exercise 7C

Where necessary leave answers approximated to 3 significant figures.

① Work out the length x in each of these triangles.

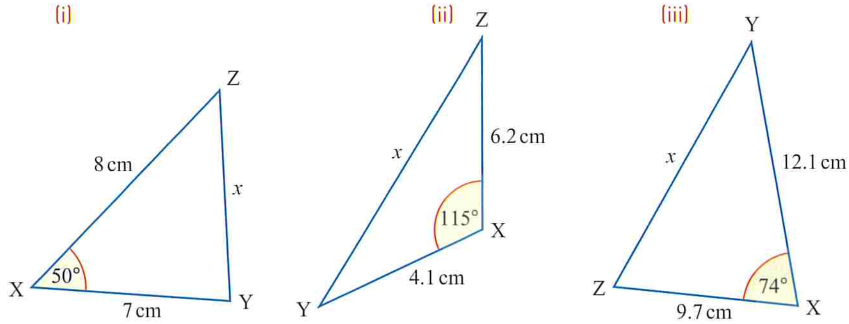


Figure 7.19

② Work out the size of the angle θ in each of the following triangles.

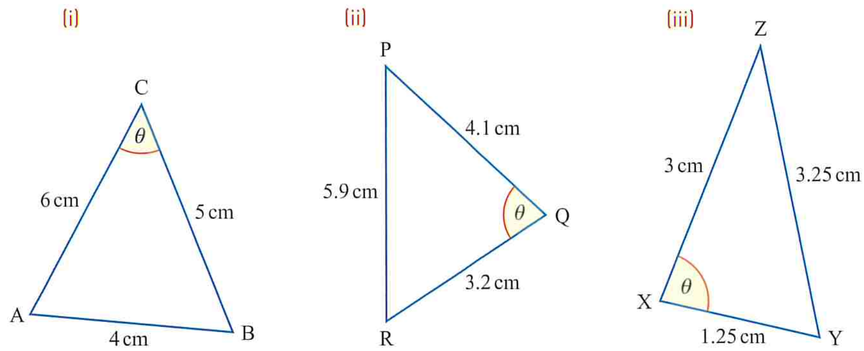


Figure 7.20

③ The diagonals of a parallelogram have lengths of 12 cm and 18 cm and the angle between them is 72°. Work out the lengths of the sides of the parallelogram.

④ Figure 7.21 shows a quadrilateral ABCD with AB = 8 cm, BC = 6 cm, CD = 7 cm, DA = 5 cm and ∠ABC = 90°.
Calculate

(i) AC

(ii) ∠ADC.

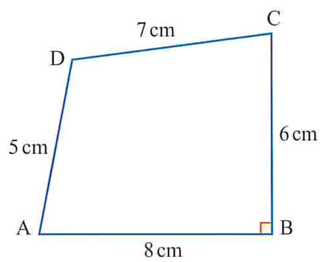


Figure 7.21

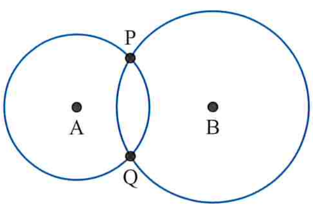


Figure 7.22

⑤ Figure 7.22 shows two circles. One has centre A and a radius of 8 cm. The other has centre B and a radius of 10 cm. AB = 12 cm and the circles intersect at P and Q. Calculate ∠PAB.

⑥ A parallelogram has sides of length 5 cm and 10 cm, and an angle of 130°.
Work out the length of the longest diagonal.

⑦ A triangle has sides of length 6 cm, 7 cm and 11 cm.
Work out its area.

⑧ Alan walks 7 km on a bearing of 054°.

He then walks 5 km on a bearing of 122°.

How far is he from his starting point?

⑨ In triangle ABC, AB = 8 cm, BC = 5 cm and ∠BAC = 35°.

Use the cosine rule to work out the possible lengths of AC.

4 Using the sine and cosine rules together

Note

Three angles are not independent measurements, as a third angle can be calculated from the other two.

When solving any triangle, three independent measurements are required.

Given **3 side-lengths**, use the cosine rule to work out the size of an angle.

Given **2 side-lengths and an included angle**, use the cosine rule to work out the length of the third side.

Given **2 side-lengths and an angle (not included)**, use the sine rule to work out the size of another angle from which you can calculate the included angle, and you can then use the cosine rule to work out the missing length. This situation can sometimes produce two possible solutions.

Given **2 angles and one side-length**, use the sine rule to work out another side-length. If the given side-length is between the two angles, then first calculate the size of the third angle using the angle sum of a triangle.

Once a fourth independent measurement of a triangle has been calculated, then the other two can be calculated using either the cosine rule or the sine rule.

Example 7.7

Figure 7.23 shows the positions of three towns, Aldbury, Bentham and Chorton.

Bentham is 8 km from Aldbury on a bearing of 037° and Chorton is 9 km from Bentham on a bearing of 150°. Work out

(i) the size of the angle ABC

(ii) the distance of Chorton from Aldbury (to the nearest 0.1 km)

(iii) the bearing of Chorton from Aldbury (to the nearest 1°).

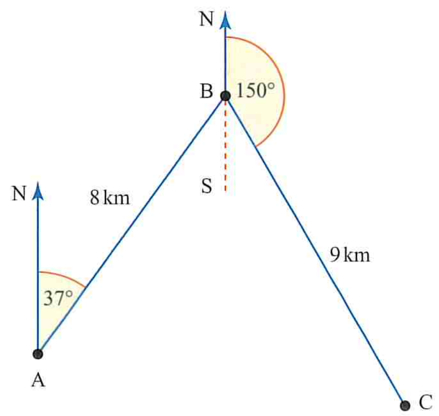


Figure 7.23

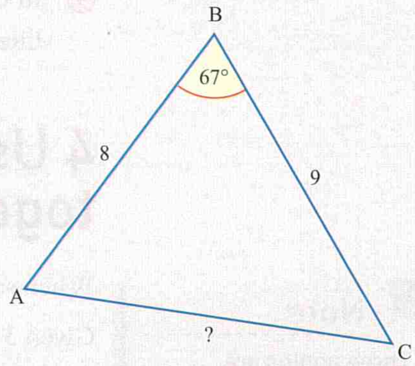
Solution

(i) $\angle ABS = 37°$ (alternate angles)
and $\angle SBC = 30°$ (adjacent angles on a straight line)
so $\angle ABC = 67°$

(ii) Using the cosine rule

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$= 9^2 + 8^2 - 2 \times 9 \times 8\cos 67°$$
$$= 88.7347\ldots$$
$$b = 9.4199\ldots$$

Chorton is 9.4 km (1 d.p.) from Aldbury.



Don't clear this from your calculator as you will need it later.

(iii) Using the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin A}{9} = \frac{\sin 67°}{9.4199\ldots}$$
$$\sin A = 0.87947\ldots$$
$$A = 61.57\ldots°$$

The bearing of Chorton from Aldbury is 099°.

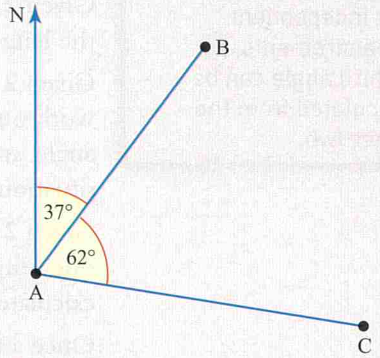


Figure 7.24

Discussion point

→ The other value of A that gives $\sin A = 0.87947\ldots$ is $118.42\ldots°$
Why does this not give an alternative solution to this problem?

Example 7.8

A triangular plot of land has sides of length 70 m, 80 m and 95 m.

Work out its area in hectares. (1 hectare is 10 000 m^2.)

Solution

First draw a sketch and label the sides.

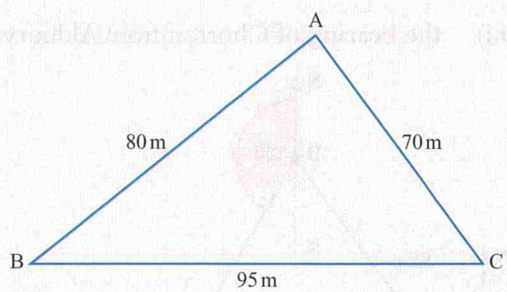


Figure 7.25

You can now see that the first step is to work out the size of one of the angles, and this will need the cosine rule.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{70^2 + 80^2 - 95^2}{2 \times 70 \times 80}$$

$$= \frac{13}{64}$$

$$\Rightarrow \quad A = 78.28°$$

$$\text{Area} = \frac{1}{2} bc \sin A$$

$$\text{Area} = \frac{1}{2} \times 70 \times 80 \times \sin 78.28°$$

$$= 2741 \text{m}^2$$

$$= 0.27 \text{ hectares (2 d.p.)}$$

Exercise 7D

Where necessary leave answers approximated to 3 significant figures.

① The hands of a clock have lengths 6 cm and 8 cm.

Work out the distance between the tips of the hands at 8 p.m.

② From a lighthouse L, a ship A is 4 km away on a bearing of 340° and a ship B is 5 km away on a bearing of 065°.

Work out the distance AB.

③ When I am at a point X, the angle of elevation of the top (T) of a vertical tree is 27°, but if I walk 20 m towards the tree along horizontal ground, to point Y, the angle of elevation is then 47°.

(i) Work out the distance TY.
(ii) Work out the height of the tree.

④ Two adjacent sides of a parallelogram have lengths 9.3 cm and 7.2 cm, and the shorter diagonal is of length 8.1 cm.

(i) Work out the sizes of the angles of the parallelogram.
(ii) Work out the length of the other diagonal of the parallelogram.

⑤ A yacht sets off from A and sails for 5 km on a bearing of 067° to a point B so that it can clear the headland before it turns onto a bearing of 146°. It then stays on that course for 8 km until it reaches a point C.

(i) Work out the distance AC.
(ii) Work out the bearing of C from A.

⑥ Two ships leave the docks, D, at the same time. *Princess Pearl*, P, sails on a bearing of 160° at a speed of 18 km h^{-1}, and *Regal Rose*, R, sails on a bearing of 105°. After 2 hours the angle DRP is 80°.

(i) Work out the distance between the ships at this time.
(ii) Work out the speed of the *Regal Rose*.

⑦ The diagram in Figure 7.26 represents a simplified drawing of the timber cross-section of a roof.

(i) Work out the lengths of the struts BD and EG.

(ii) Work out the length DE.

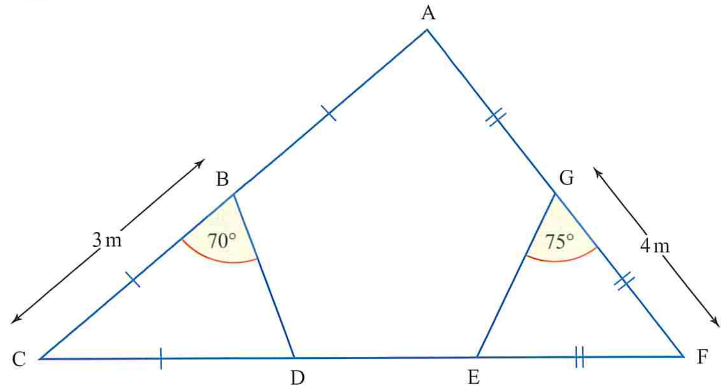


Figure 7.26

⑧ Sam and Aziz cycle home from school.

Sam cycles due east for 4 km, and Aziz cycles due south for 3 km and then for 2 km on a bearing of 125°.

How far apart are their homes?

5 Problems in three dimensions

Discussion point

→ An aircraft flying between two places at the same latitude doesn't usually follow a route along the line of latitude. Why?

Figure 7.27

When you are solving three-dimensional problems it is important to draw good diagrams (although you will not be assessed on this in the exam, a clear diagram does benefit understanding). There are two types:

- representations of three-dimensional objects
- true shape diagrams of two-dimensional sections within a three-dimensional object.

Representations of three-dimensional objects

Figures 7.28 and 7.29 illustrate ways in which you can draw a clear diagram.

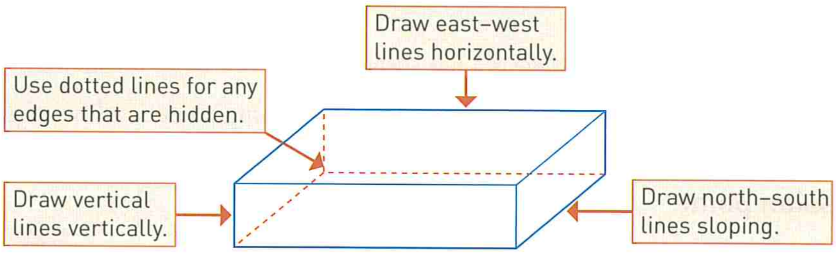


Figure 7.28

Open up the diagram as much as possible by choosing a suitable direction for your *north–south* axis; (a) is clearer than (b).

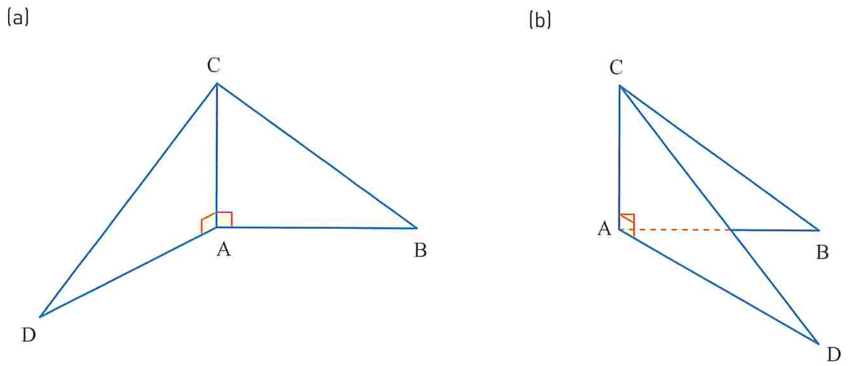


Figure 7.29

True shape diagrams

In a two-dimensional representation of a three-dimensional object, right angles do not always appear to be 90°, so draw as many true shape diagrams as necessary.

For example, if you need to do calculations on the triangular cross-section BCD in Figure 7.30(a), you should draw the triangle so that the right angle really does look 90° as in Figure 7.30(b).

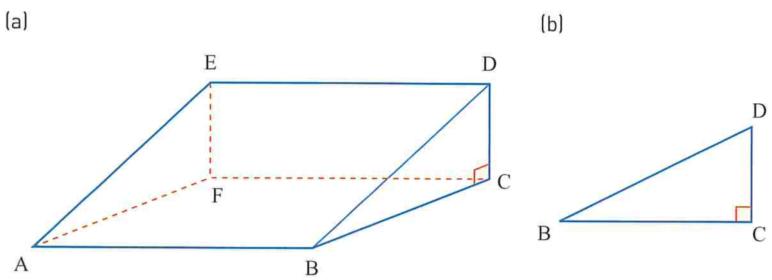


Figure 7.30

6 Lines and planes in three dimensions

A *plane* is a flat surface (not necessarily horizontal).

A *line of greatest slope* of a sloping plane is a line of greatest gradient, i.e. the line that a ball would follow if allowed to roll down it. This is shown in Figure 7.31.

Discussion point

→ Give an example of a sloping plane from everyday life.

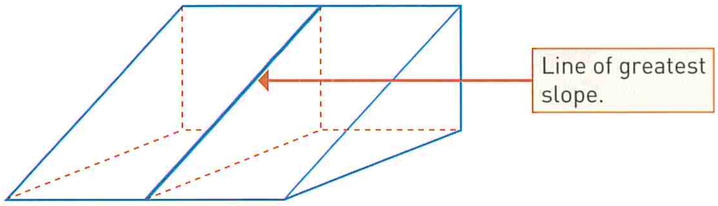


Figure 7.31

In three-dimensional problems you need to be aware of the relationships between lines and planes.

Two lines

In two dimensions, two lines either meet (when extended if necessary), or they are parallel.

In three dimensions, there is a third option: they are *skew*, as in Figure 7.32.

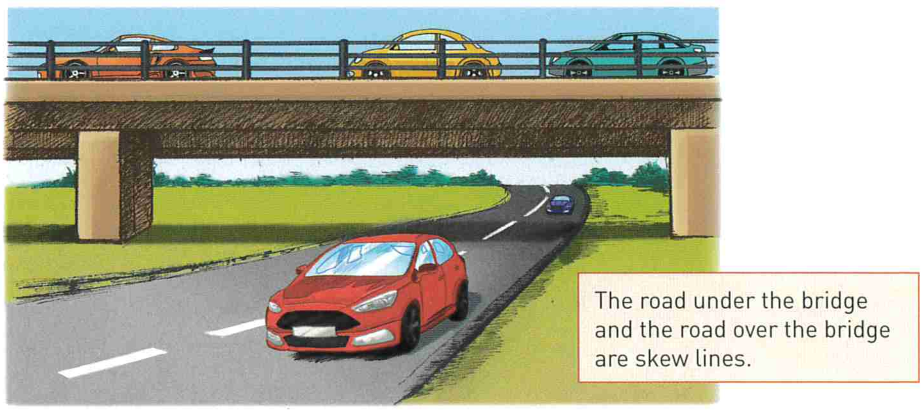


Figure 7.32

A line and a plane

In three dimensions there are three options, as shown in Figure 7.33.

(a) The line and the plane are *parallel*. A curtain rail is *parallel* to the floor.

(b) The line meets the plane at a *single point*. When you are writing, your pen meets the paper at a *single point*.

(c) The line *lies in* the plane. When you put your pen down, your pen *lies in* the plane of the paper.

(a) (b) (c)

Figure 7.33

Angle between a line and a plane

Draw a perpendicular from the line to the plane.

Line PQ meets the plane ABCD at Q.

PR is perpendicular to the plane.

QR is in the plane.

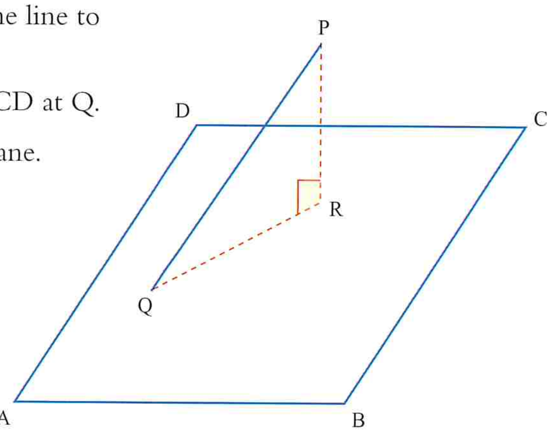


Figure 7.34

Angle between line and plane is angle PQR.

Discussion point

➜ Give other examples of these cases.

Two planes

In three dimensions there are two options.

(a) The two planes are parallel. Opposite walls of a room are usually parallel.

(b) The two planes meet *in a line*. The ceiling meets each wall of a room *in a line*. An open gate and a wall meet *in a line*.

(a) (b)

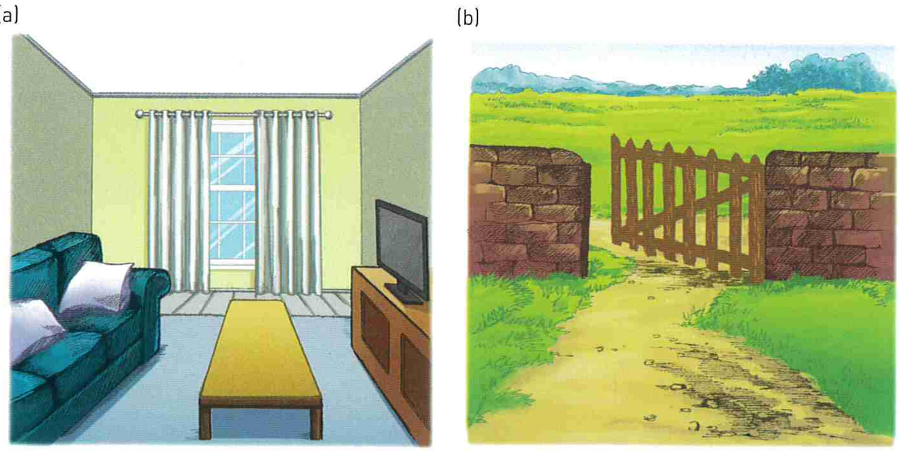


Figure 7.35

Angle between two planes

Identify the line where the planes meet.

Draw a line in each plane that is perpendicular to the line where the planes meet.

The angle between these two lines is the angle between the planes.

Planes ABCD and APQD meet along AD.

The dashed lines are each perpendicular to AD.

x is the angle between the planes.

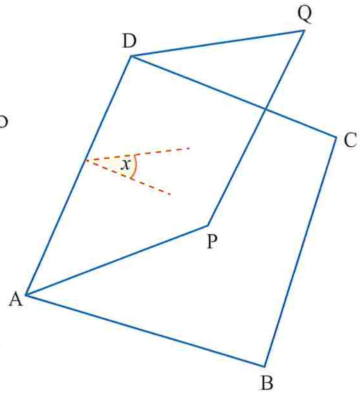


Figure 7.36

Example 7.9

Figure 7.37 shows a wedge ABCDEF with AB = 8 cm, BC = 6 cm and CD = 2 cm. The angle BCD is 90°.

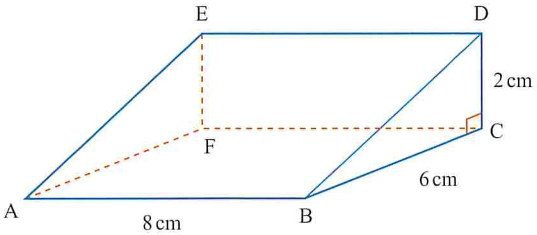


Figure 7.37

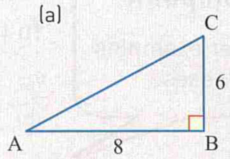
Work out

(i) the length AC (ii) the length AD

(iii) the size of the angle between DA and ABCF

(iv) the size of the angle between ABDE and ABCF.

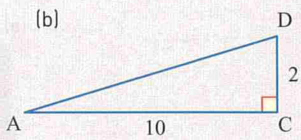
Solution



(i) From Figure 7.38(a)

$AC^2 = 8^2 + 6^2$ (Pythagoras)

$\Rightarrow \quad AC = 10\text{ cm}$



(ii) From Figure 7.38(b)

$AD^2 = AC^2 + 2^2$ (Pythagoras)

$\Rightarrow \quad AD = 10.2\text{ cm}$ (1 d.p.)

DA and ABCF meet at A. DC is perpendicular to ABCF.

(iii) From Figure 7.38(b), the angle between DA and ABCF is ∠DAC.

$\tan \angle DAC = \dfrac{2}{10}$

$\Rightarrow \angle DAC = 11.3°$ (1 d.p.)

ABDE and ABCF meet along AB. BD is perpendicular to AB. BC is perpendicular to AB.

(iv) From Figure 7.38(c), the angle between ABDE and ABCF is ∠DBC

$\tan \angle DBC = \dfrac{2}{6}$

$\Rightarrow \quad \angle DBC = 18.4°$ (1 d.p.)

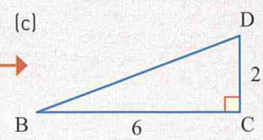


Figure 7.38

Example 7.10

RWC

Figure 7.39 shows a straight level road AB, 400 m long. A vertical radio mast XY stands some distance from the road, and the bottom of the mast, X, is on the same level as the road. The angle of elevation of Y from A is 30°, ∠XAB = 25° and ∠AXB = 90°. Calculate

(i) the distance AX

(ii) the height of the mast

(iii) the distance of X from the road.

Give your answers to 3 significant figures.

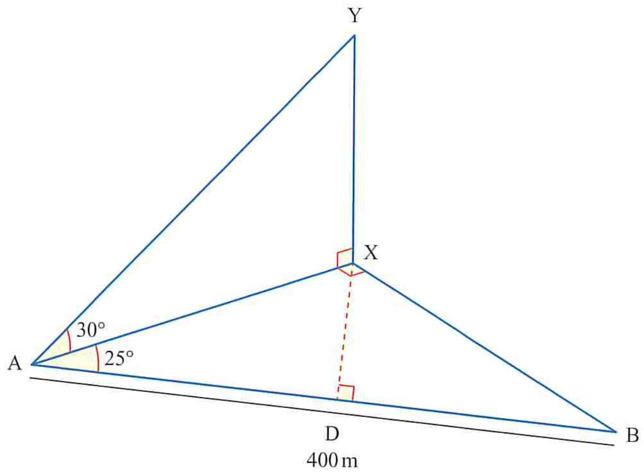


Figure 7.39

Solution

(i) From Figure 7.40(a)

$$\frac{AX}{400} = \cos 25°$$

⇒ AX = 362.523…

⇒ The distance AX = 363 m.

(ii) From Figure 7.40(b)

$$\frac{XY}{362.523\ldots} = \tan 30°$$

⇒ XY = 209.302…

⇒ The height of the mast XY = 209 m.

(iii) From Figure 7.40(c)

$$\frac{DX}{362.523\ldots} = \sin 25°$$

⇒ DX = 153.208…

⇒ The distance of X from the road = 153 m.

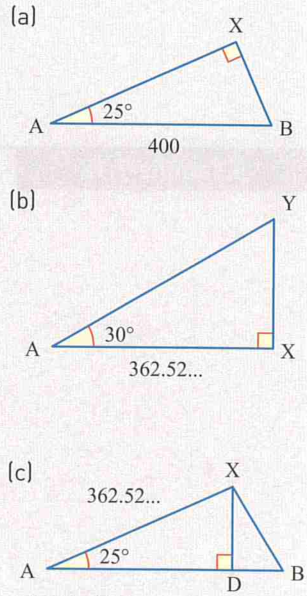


Figure 7.40

Example 7.11

The pyramid VABCD has square horizontal base ABCD.

The vertex, V, is directly above the centre, X, of the base.

M is the midpoint of BC.

AB = 8 metres and VX = 15 metres.

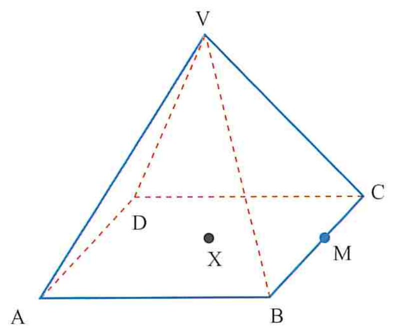


Figure 7.41

Work out the angle between the planes ABCD and VBC.

Solution

The planes meet along BC.

MX and VM are both perpendicular to BC.

Angle VXM is 90°.

XM = 8 ÷ 2

= 4 m

$\tan \text{VMX} = \frac{15}{4}$

angle VMX = 75.1° (1 d.p.)

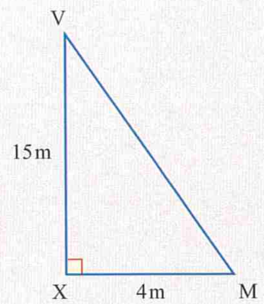


Figure 7.42

Example 7.12

The cuboid has a square base ABCD of side 8 cm and a height of 4 cm.

M is the midpoint of AC.

(i) Calculate the exact length of DM.

(ii) Work out the angle between the planes ABCD and ACH.

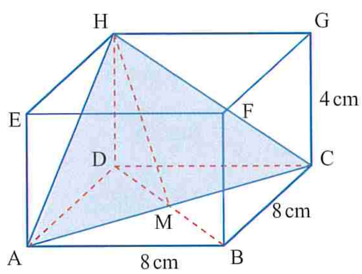


Figure 7.43

Solution

(i) DM = half the length of the diagonal of the square base

$= \frac{1}{2}\sqrt{8^2 + 8^2}$

$4\sqrt{2}$ cm

(ii) The angle required is ∠HMD.

$\tan \text{HMD} = \frac{\text{HD}}{\text{DM}}$

$= \frac{4}{4\sqrt{2}}$

The required angle is 35.3°.

Pythagoras' theorem in three dimensions

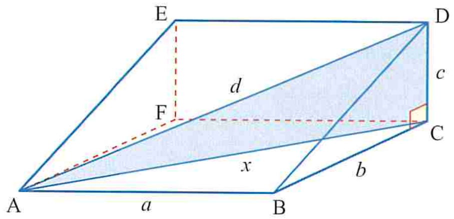


Figure 7.44

ACTIVITY 7.2

Look for integer values of a, b, c and d such that $a^2 + b^2 + c^2 = d^2$

You could start with $3^2 + 4^2 = 5^2$ and then use $5^2 + 12^2 = 13^2$

Can you find at least two examples of values of a, b, c and d?

In Figure 7.44, the base is rectangular, so using Pythagoras' theorem in 2 dimensions

$a^2 + b^2 = x^2$

The triangle ACD has a right angle at C, giving

$x^2 + c^2 = d^2$

Substituting for x^2 from the first equation gives

$a^2 + b^2 + c^2 = d^2$

This is the 3-D version of Pythagoras' theorem.

Example 7.13

ABCDEFGH is a cuboid with side-lengths as shown in the diagram.

Calculate the length of the diagonal AF.

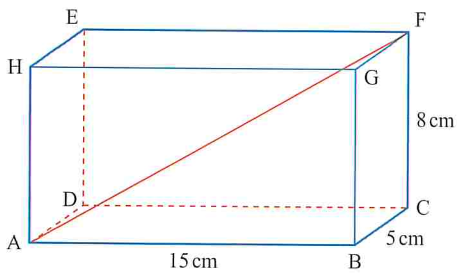


Figure 7.45

Solution

$\text{AF} = \sqrt{15^2 + 5^2 + 8^2}$

$= \sqrt{314}$

$= 17.7$ cm (3 s.f.)

Use of the sine and cosine rules in 3-D problems

Example 7.14

ABCDEFGH is a cuboid with side-lengths as shown in the diagram.

(i) Calculate the size of angle HDF.

(ii) Hence, calculate the size of angle DHF.

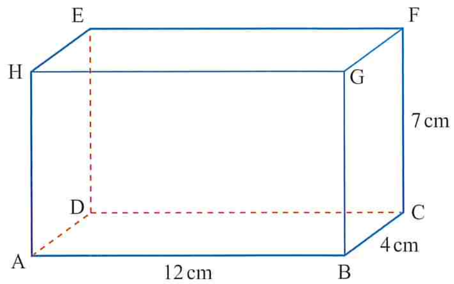


Figure 7.46

Solution

Consider triangle HDF.

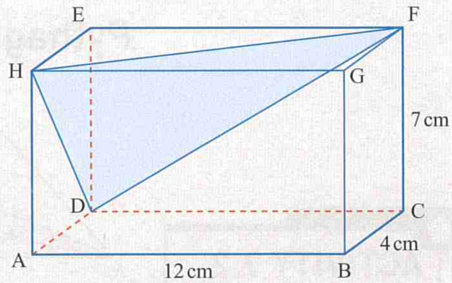


Figure 7.47

(i) $HD = \sqrt{4^2 + 7^2} = \sqrt{65}$

$FD = \sqrt{12^2 + 7^2} = \sqrt{193}$

$HF = \sqrt{4^2 + 12^2} = \sqrt{160}$

Using the cosine rule: $\cos HDF = \dfrac{HD^2 + FD^2 - HF^2}{2 \times HD \times FD}$

$\cos HDF = \dfrac{65 + 193 - 160}{2 \times \sqrt{65} \times \sqrt{193}}$

$\cos HDF = 0.437$

$HDF = 64.1° \ (1 \text{ d.p.})$

(ii) Using the sine rule: $\dfrac{\sin H}{FD} = \dfrac{\sin D}{HF}$

$\dfrac{\sin H}{\sqrt{193}} = \dfrac{\sin 64.1}{\sqrt{160}}$

$\sin H = 0.988$

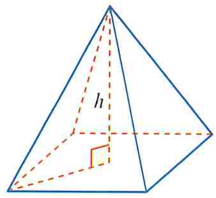
$DHF = 81.0° \ (1 \text{ d.p.})$

Pyramids and cones

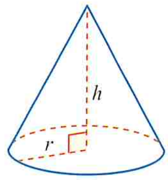
Pyramids and cones are three-dimensional shapes. Although a cone is not a pyramid, it has the same properties. Pyramids must have a polygonal base.

In both cases their volumes are given by the formula

$$V = \frac{1}{3} \times \text{base area} \times \text{height}$$



Volume = $\frac{1}{3}$ × base area × height



Volume = $\frac{1}{3}$ × base area × height = $\frac{1}{3}\pi r^2 h$

Figure 7.48

Example 7.15

A cone with a base radius of 6 cm and a height of 3π cm has the same volume as a pyramid with a square base of side 2π cm. What is the height of the pyramid?

Solution

Let h = height of pyramid

Volume of pyramid = Volume of cone

$$\therefore \quad \frac{1}{3}(2\pi)^2 h = \frac{1}{3}\pi 6^2 \times 3\pi$$

$$\Rightarrow \quad \frac{4\pi^2 h}{3} = \frac{108\pi^2}{3}$$

$$\Rightarrow \quad 4h = 108$$

$$\Rightarrow \quad h = 27\text{ cm}$$

Exercise 7E

① The cube ABCDEFGH shown in the diagram has sides of length 10 cm.

Calculate

(i) the length AC
(ii) the length AG
(iii) the angle GAC.

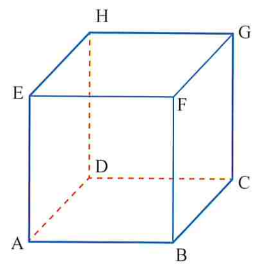


Figure 7.49

② Figure 7.50 represents a pyramid ABCD with a horizontal base ABC.

AB = AC = 5 cm and BD = CD = 13 cm.

D is vertically above A and ∠BAD = ∠CAD = 90°.

M is the midpoint of BC.

Calculate

(i) the length AM
(ii) the angle BCD
(iii) the angle between the planes BCA and BCD.

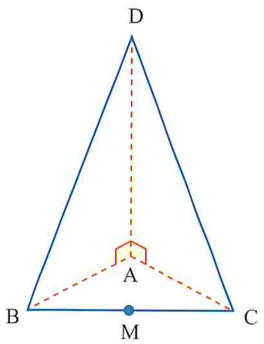


Figure 7.50

RWC ③ Figure 7.51 shows a wedge ABCDEF which has been made to hold a door open.

AB = 5 cm, BC = 12 cm and FC = 4 cm.

Calculate

(i) the angle FBC

(ii) the length AC

(iii) the angle between the line FA and the plane ABCD.

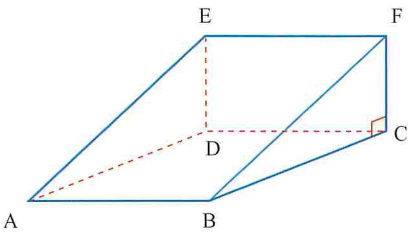


Figure 7.51

There is a gap of 2 cm between the door and the floor.

(iv) How far along BF will the base of the door meet the wedge?

④ A, B and C are points on a horizontal plane.

A is 75 m from C on a bearing of 210° and the bearing of B from C is 120°. The bearing of B from A is 075°.

From A, the angle of elevation of the top T of a vertical tower at C is 42°.

Calculate

(i) the distance BC

(ii) the height of the tower

(iii) the angle of elevation of T from B.

RWC ⑤ C is the foot of a vertical tower CT 28 m high.

A and B are points in the same horizontal plane as C and CA = CB.

P is the point on AB that is nearest to C.

The angle of elevation of the top of the tower from P is 40° and ∠ACB = 120°.

Calculate

(i) the length CP

(ii) the length CB

(iii) the length AB

(iv) the angle of elevation of the top of the tower from B.

RWC ⑥ The waste-paper basket shown in Figure 7.52 has a top ABCD that is a square of side 30 cm and a base PQRS that is a square of side 20 cm.

The line joining the centres of the top and base is perpendicular to both and is 40 cm long.

Calculate

(i) the length PR

(ii) the length AC

(iii) the length AP.

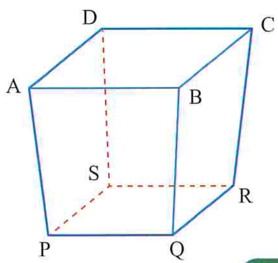


Figure 7.52

RWC ⑦ In Egypt, pyramids were used as burial chambers for the Pharaohs.

The largest of these, shown in the diagram and built about 2500 BC for Cheops, is 146 m high and has a square base of side 231 m.

X is the centre of the base and VX = 146 m.

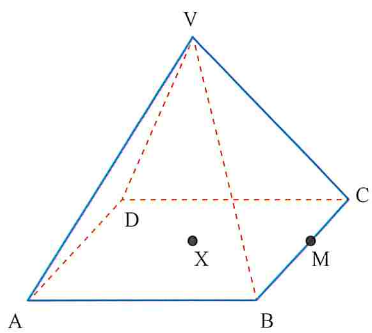


Figure 7.53

Calculate

(i) the angle between VA and ABCD
(ii) the length VA
(iii) the length VM where M is the midpoint of AB
(iv) the angle between VAB and ABCD.

RWC ⑧ The tent shown in Figure 7.54 has a base that is 2.2 m wide and 3.6 m long. The ends are isosceles triangles, inclined at an angle of 80° to the base. ∠AEB = ∠DFC = 70° and M is the midpoint of AB.

Calculate

(i) the length of EM
(ii) the height of EF above the base
(iii) the length of EF.

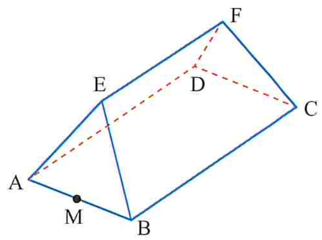


Figure 7.54

⑨ The right pyramid VABCD has rectangular base ABCD. The vertex, V, is directly above the centre, X, of the base. M is the midpoint of BC.

AB = 12 metres, BC = 9 metres and VA = 18 metres.

Work out

(i) the length AC
(ii) the length VX
(iii) the angle between VA and ABCD
(iv) the angle between VBC and ABCD.

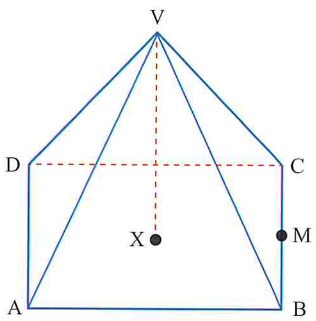


Figure 7.55

RWC ⑩ A new perfume is to be packaged in a box that is in the shape of a regular tetrahedron VABC of side 6 cm standing on a triangular prism ABCDEF as shown in the diagram.

The height of the prism is 12 cm.

M is the midpoint of BC.

Calculate

(i) the length AM
(ii) the length VM
(iii) the angle VAM
(iv) the total height of the box.

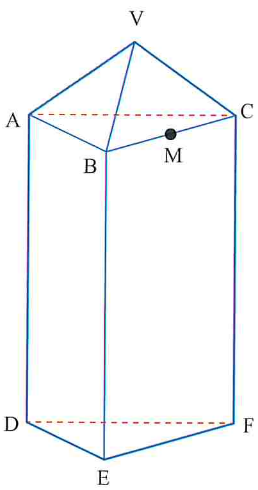


Figure 7.56

⑪ The cuboid has a square base ABCD of side 6 cm and a height of 3 cm. M is the midpoint of EG.

(i) Calculate the length of BM.
(ii) Work out the area of triangle BEG.
(iii) Work out the angle between triangle BEG and the plane ABCD.

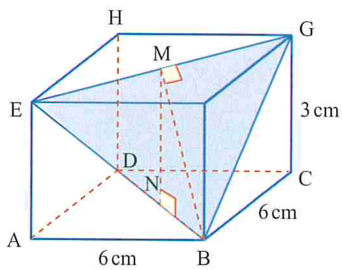


Figure 7.57

⑫ The cube has sides of 12 cm and M is the midpoint of AC.

(i) Calculate the length of DM.
(ii) Work out the angle between the planes ABCD and ACH.
(iii) Calculate the area of the largest triangle that would fit inside this cube.
(iv) What is the area of the largest triangle that would fit inside a cube of side 20 cm? Give your answer in exact form.

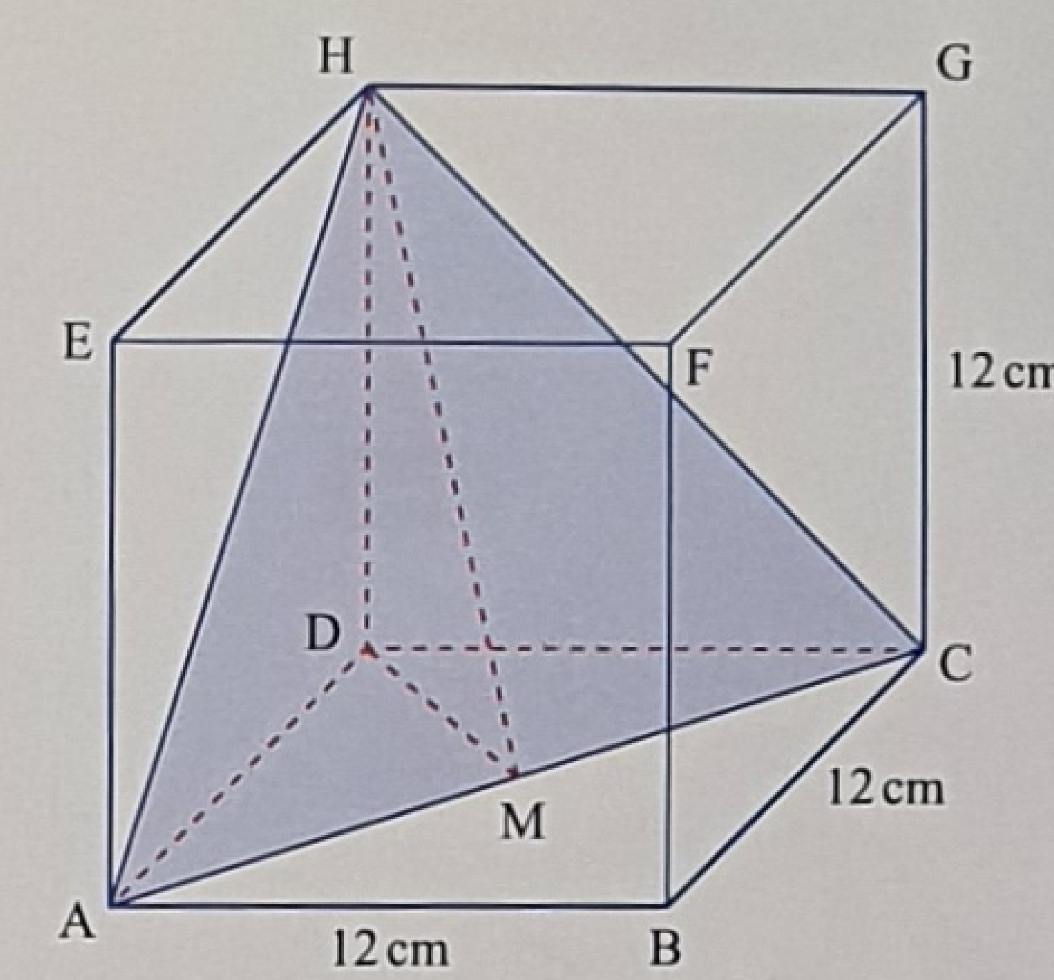


Figure 7.58

⑬ A cuboid ABCDEFGH has edges of length 8 cm, 3 cm and 5 cm as shown.

Calculate the size of the smallest angle in triangle AEG.

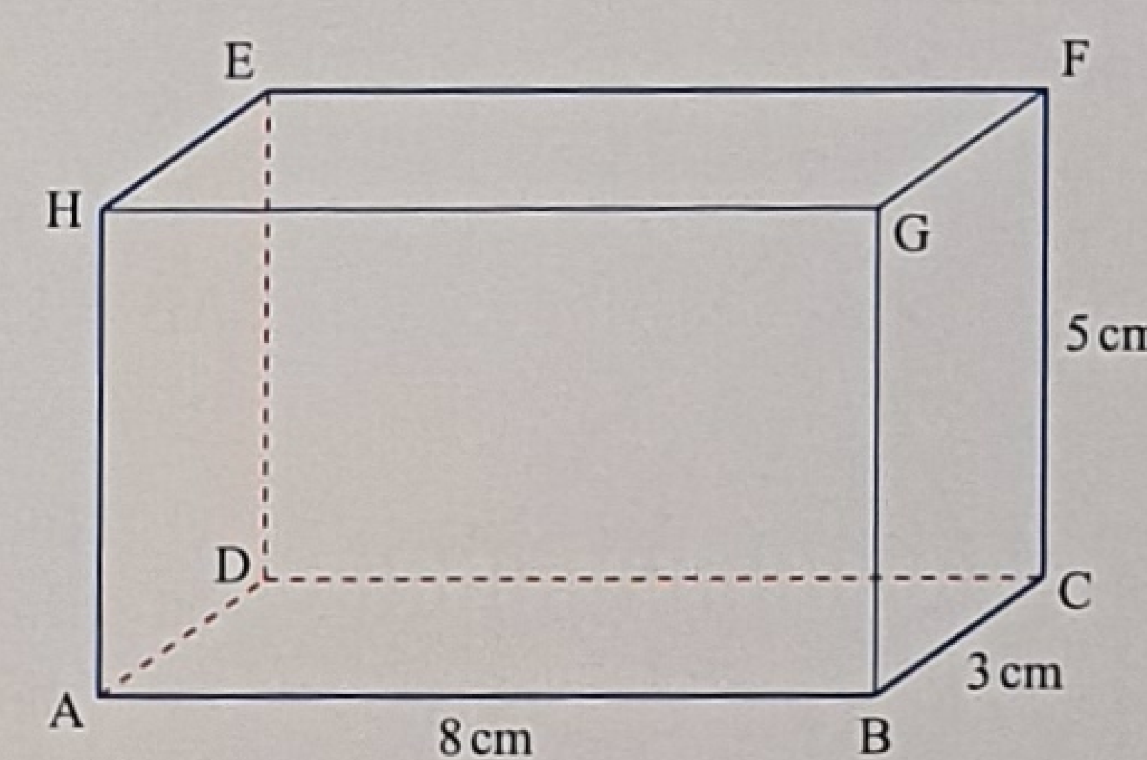


Figure 7.59

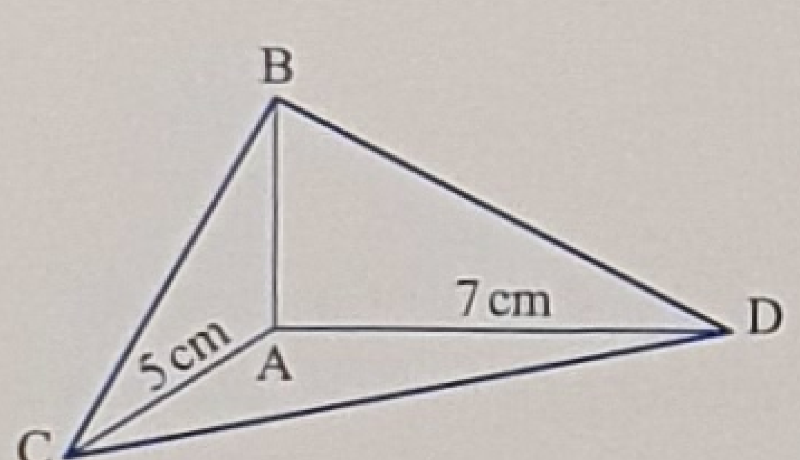


Figure 7.60

⑭ A tetrahedron ABCD has lengths AC = 5 cm and AD = 7 cm. Given that angle CAD = 130°, angle BCD = 65° and angle BDC = 55°, calculate the length of edge BC.

⑮ Tetrahedron PQRS has lengths PQ = 8 cm, PS = 10 cm and QR = 5 cm, and angles QPS = 64° and QRS = 73°. Calculate the size of angle QSR.

FUTURE USES

You will use the principles introduced here in the study of lines and planes in vector form at A-Level.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- calculate the area of a triangle given two sides and an included angle
- use the sine rule to calculate the size of an angle or a side-length
- use the cosine rule to calculate the size of an angle or side-length
- draw a 2-D representation of a 3-D object
- calculate the angle between a line and a plane or the angle between two planes
- use Pythagoras' theorem to calculate lengths in three dimensions
- solve practical problems in three dimensions using the knowledge above.

REAL-WORLD CONTEXT

Angles between two lines, a line and a plane or two planes are of prime importance to architects and engineers in the design of buildings and machinery.

There are applications in navigation, for both ships and aircraft.

There are also applications in software engineering.

KEY POINTS

1 Area of a triangle $= \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

2 Sine rule : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ and $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

3 Cosine rule : $a^2 = b^2 + c^2 - 2bc\cos A$ and $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

4 When solving three-dimensional problems always draw a clear diagram where:

- vertical lines are drawn vertically
- east–west lines are drawn horizontally
- north–south lines are drawn sloping
- edges that are hidden are drawn as dotted lines.

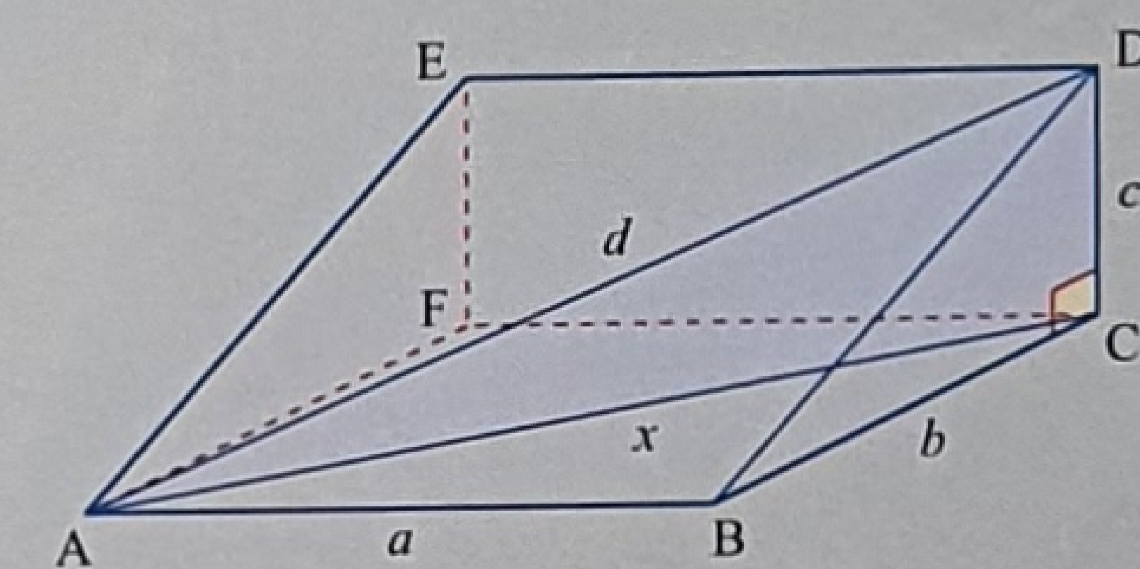


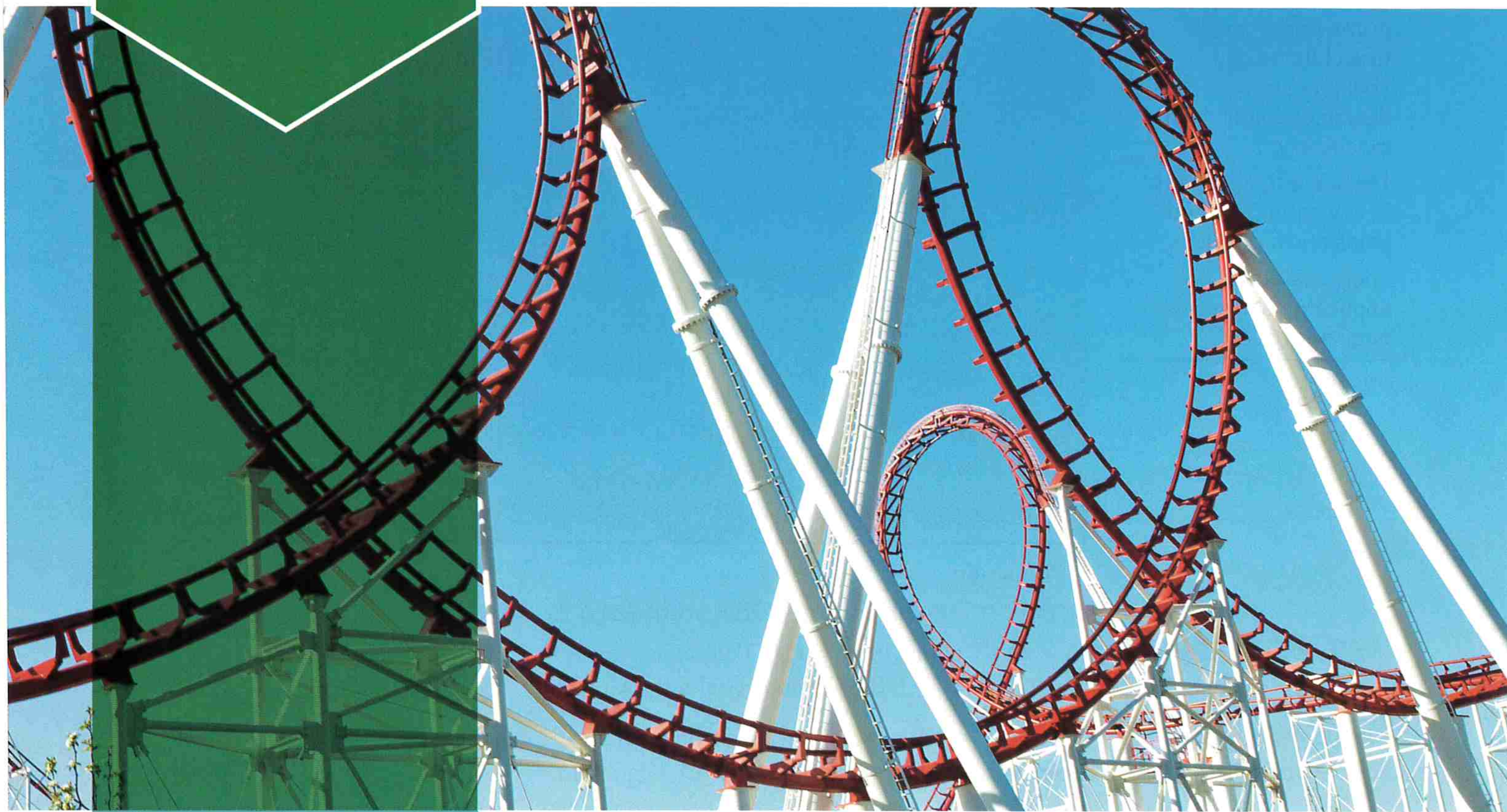
Figure 7.61

5 In three dimensions, Pythagoras' theorem extends to

$a^2 + b^2 + c^2 = d^2$

6 Volume of a pyramid (or cone) $= \frac{1}{3} \times$ base area $\times$ perpendicular height

8 Calculus



I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Isaac Newton

Prior knowledge

The formula 'gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$' was introduced in Chapter 3 for the gradient of a straight line joining the two points (x_1, y_1) and (x_2, y_2).

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ leads to a general equation of a straight line $(y - y_1) = m(x - x_1)$.

1 The gradient of a curve

In Figure 8.1 the curve has a zero gradient at A, a positive gradient at B and a negative gradient at C.

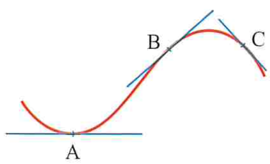
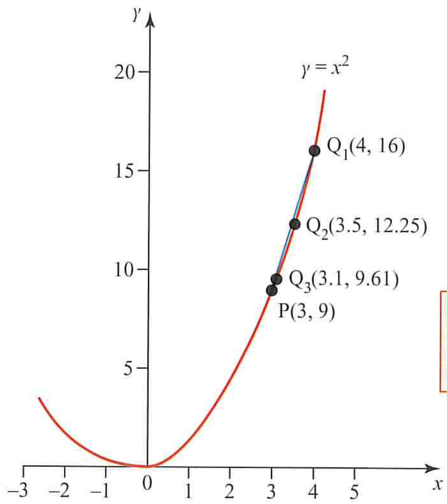


Figure 8.1

One way of finding these gradients is to draw the tangents and use two points on each one to calculate its gradient. This is time-consuming and the results depend on the accuracy of your drawing and measuring. If you know the equation of the curve, then *differentiation* provides another method of calculating the gradient.

2 Differentiation

Instead of trying to draw an accurate tangent, this method starts by calculating the gradients of chords PQ_1, PQ_2, … . As the different positions of Q get closer to P, the values of the gradient of PQ get closer to the gradient of the tangent at P. The first few positions of Q are shown in Figure 8.2.



The line PQ_1 joining two points on the curve is called a chord.

Figure 8.2

For P at (3, 9)

chord	coordinates of Q	gradient of PQ
PQ_1	(4, 16)	$\dfrac{16-9}{4-3} = 7$
PQ_2	(3.5, 12.25)	$\dfrac{12.25-9}{3.5-3} = 6.5$
PQ_3	(3.1, 9.61)	$\dfrac{9.61-9}{3.1-3} = 6.1$
PQ_4	(3.01, 9.0601)	$\dfrac{9.0601-9}{3.01-3} = 6.01$
PQ_5	(3.001, 9.006 001)	$\dfrac{9.006\,001-9}{3.001-3} = 6.001$

ACTIVITY 8.1

Take points R_1 to R_5 on the curve $y = x^2$ with x-coordinates 2, 2.5, 2.9, 2.99, and 2.999 respectively and work out the gradients of the chords joining each of these points to P(3, 9).

In this process the gradient of the chord PQ gets closer and closer to that of the tangent, and hence the gradient of the curve at (3, 9).

Look at the sequence formed by the gradients of the chords.

7, 6.5, 6.1, 6.01, 6.001, …

It looks as though this sequence is converging to 6

The table and the activity show that the gradient of the curve $y = x^2$ at (3, 9) seems to be 6 or about 6 but do not provide conclusive proof of its value. To do that you need to apply the method in more general terms.

Take the point P(3, 9) and another point Q close to (3, 9) on the curve $y = x^2$. Let the x-coordinate of Q be $(3 + h)$ where h is small. Since $y = x^2$ at all points on the curve, the y-coordinate of Q will be $(3 + h)^2$.

Figure 8.3 shows Q in a position where h is positive. Negative values of h would put Q to the left of P.

From Figure 8.3, the gradient of PQ is $\dfrac{(3+h)^2 - 9}{h}$

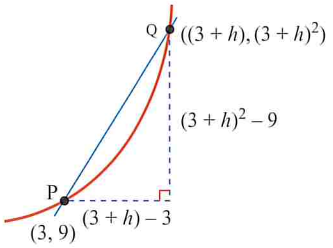


Figure 8.3

$$= \frac{9 + 6h + h^2 - 9}{h}$$
$$= \frac{6h + h^2}{h}$$
$$= \frac{h(6+h)}{h}$$
$$= 6 + h.$$

For example, when $h = 0.001$, the gradient of PQ is 6.001 and when $h = -0.001$, the gradient of PQ is 5.999. The gradient of the tangent at P is between these two values. Similarly the gradient of the tangent at P would be between $6 - h$ and $6 + h$ for all small non-zero values of h.

For this to be true, the gradient of the tangent at (3, 9) must be *exactly* 6.

In this case, 6 was the *limit* of the gradient values, whether you approached P from the right or the left.

ACTIVITY 8.2

Using a similar method, work out the gradient of the tangent to the curve at

(i) (2, 4)

(ii) (−1, 1)

(iii) (−3, 9).

What do you notice?

The gradient function

The work so far has involved calculating the gradient of the curve $y = x^2$ at just one particular point. It would be very tedious if you had to do this every time and so instead you can consider a general point (x, y) and then substitute the value(s) of x and/or y corresponding to the point(s) of interest. The gradient function is a measure of how the function is changing – often referred to as 'the rate of change of the function'.

Example 8.1

Calculate the gradient of the curve $y = x^3$ at the general point (x, y).

Solution

Let P have the general value x as its x-coordinate, so P is the point (x, x^3) (since it is on the curve $y = x^3$).

Let the x-coordinate of Q be $(x + h)$ so Q is the point $((x + h), (x + h)^3)$.

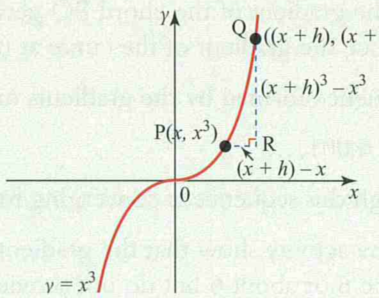


Figure 8.4

The gradient of the chord PQ is given by

$$\frac{QR}{PR} = \frac{(x+h)^3 - x^3}{(x+h) - x}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \frac{h\left(3x^2 + 3xh + h^2\right)}{h}$$

$$= 3x^2 + 3xh + h^2$$

As Q gets closer to P, h takes smaller and smaller values and the gradient approaches the value of $3x^2$, which is the gradient of the tangent at P.

The gradient of the curve $y = x^3$ at the point (x, y) is equal to $3x^2$.

ACTIVITY 8.3

Use the method in Example 8.1 to prove that the gradient of the curve $y = x^4$ at the point (x, y) is equal to $4x^3$.

An alternative notation

So far, h has been used to denote the difference between the x-coordinates of our points P and Q, where Q is close to P.

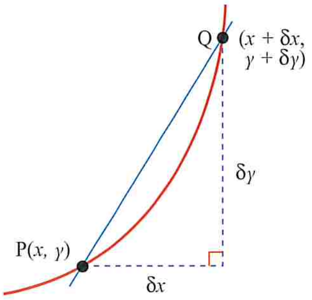


Figure 8.5

h is sometimes replaced by δx. The Greek letter δ (delta) is shorthand for 'a small change in' and so δx represents a small change in x, δy a small change in y and so on.

In Figure 8.5 the gradient of the chord PQ is $\frac{\delta y}{\delta x}$.

In the limit as δx tends towards 0, δx and δy both become infinitesimally small and the value obtained for $\frac{\delta y}{\delta x}$ approaches the gradient of the tangent at P.

Read this as 'the limit as δx tends towards 0'.

$\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$ is written as $\frac{dy}{dx}$.

Using this notation, you have a rule for differentiation.

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

The gradient function, $\frac{dy}{dx}$, is sometimes called the *derivative* of y with respect to x and when you find it you have *differentiated* y with respect to x.

Because of the connection with gradient, $\frac{dy}{dx}$ is also referred to as the rate of change of y with respect to x.

Notation

An alternative way of expressing a function such as $y = 2x^2 + x - 3$ is to replace y by $f(x)$ and write $f(x) = 2x^2 + x - 3$. When discussing a function written in this form you just say 'f of x equals two x squared plus x minus three'. $f'(x)$ is the notation used for the differential of $f(x)$, so when $y = f(x)$ you would write $\frac{dy}{dx} = f'(x)$. The derivative of $f(x)$ is written as $f'(x)$, pronounced as 'f dashed x'.

ACTIVITY 8.4

From earlier work you know that all lines of the form $y = x + c$, (where c can be positive, negative or zero) are parallel.

Using any software at your disposal, sketch graphs of $y = x^2$, $y = x^2 + 5$ and $y = x^2 - 3$ on the same axes setting your axes to $-4 < x < 4$ and $-5 < y < 25$

What do you notice?

Repeat this for the graphs of $y = x^2 + 2x$, $y = x^2 + 2x + 5$ and $y = x^2 + 2x - 3$

3 Differentiation using standard results

Finding the gradient from first principles establishes a formal basis for differentiation but in practice you would use the differentiation rule. This also includes the results obtained by differentiating (i.e. finding the gradient of) equations which represent straight lines.

The gradient of the line $y = x$ is 1

The gradient of the line $y = c$ is 0, where c is a constant, since this line is parallel to the x-axis.

The rule can be extended further to include functions of the type $y = kx^n$ for any constant k, to give

$$y = kx^n \Rightarrow \frac{dy}{dx} = nkx^{n-1}.$$

You may find it helpful to remember the rule as

'multiply by the power of x and reduce the power by 1'.

Reflecting on Activity 8.4 and using this rule:

$$y = x^2 + 2x \Rightarrow \frac{dy}{dx} = 2x + 2$$

$$y = x^2 + 2x + 5 \Rightarrow \frac{dy}{dx} = 2x + 2$$

$$y = x^2 + 2x - 3 \Rightarrow \frac{dy}{dx} = 2x + 2$$

The three graphs have the same gradient function so are parallel.

Example 8.2

Write down the gradient function for each of the following functions.

(i) $y = x^7$ (ii) $y = 4x^3$ (iii) $y = 5x^2$

Solution

(i) $\frac{dy}{dx} = 7x^6$ (ii) $\frac{dy}{dx} = 12x^2$ (iiii) $\frac{dy}{dx} = 10x$

Exactly the same rule, 'multiply by the power of x and reduce the power by 1' applies when the power is zero (i.e. y = a constant) or is negative. Remember that, for example, when you subtract 1 from 0, the answer is −1 and when you subtract 1 from −3 the answer is −4

For $y = x^0$, the rule gives $\frac{dy}{dx} = 0 \times x^{-1}$ which = 0

Example 8.3

Work out the gradient function for each of the following functions.

(i) $y = x^{-3}$ (ii) $y = 2x^{-4}$ (iii) $y = \frac{3}{x^2}$ (iv) $y = \frac{3}{4x^2}$

Solution

(i) $\frac{dy}{dx} = -3x^{-4}$

(ii) $\frac{dy}{dx} = -8x^{-5}$

(iii) First write $y = \frac{3}{x^2}$ as $y = 3x^{-2}$ $\rightarrow \frac{dy}{dx} = -6x^{-3} = -\frac{6}{x^3}$

(iv) First write $y = \frac{3}{4x^2}$ as $y = \frac{3}{4}x^{-2}$ $\rightarrow \frac{dy}{dx} = \frac{3}{4}\left(-2x^{-3}\right) = -\frac{3}{2x^3}$

You must leave the 4 in the denominator until you can simplify at the end.

Sums and differences of functions

Many of the functions you will meet are sums or differences of simpler ones. For example, the function $(4x^3 + 3x)$ is the sum of the functions $4x^3$ and $3x$. To differentiate a function such as this you differentiate each part separately and then add the results together.

Example 8.4

Differentiate $y = 4x^3 + 3x$.

Solution

$\frac{dy}{dx} = 12x^2 + 3$

Example 8.5

Differentiate $y = \frac{x^2}{2} - \frac{2}{3x^2}$.

Solution

Start by writing the expression in the form $y = \frac{1}{2}x^2 - \frac{2}{3}x^{-2}$

Differentiating, $\frac{dy}{dx} = \frac{1}{2}(2x) - \frac{2}{3}(-2x^{-3})$

$$= x + \frac{4}{3}x^{-3}$$

$$= x + \frac{4}{3x^3}$$

Example 8.6

Given that $y = 2x^3 - 3x + 4$, work out

(i) $\frac{dy}{dx}$

(ii) the gradient of the curve at the point (2, 14)

(iii) the rate of change of y with respect to x when $x = -3$

Solution

(i) $\frac{dy}{dx} = 6x^2 - 3$

(ii) At (2, 14), $x = 2$

Substituting $x = 2$ in the expression for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = 6 \times (2)^2 - 3 = 21$$

(iii) $\frac{dy}{dx}$ is the rate of change of y with respect to x.

Substituting $x = -3$ in the expression for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = 6 \times (-3)^2 - 3$$

$$= 51$$

Example 8.7

Given that $y = 5x^2 - \frac{5}{x^2} + 2$, work out

(i) $\frac{dy}{dx}$

(ii) the gradient of the curve at the point (1, 2)

(iii) the rate of change of y with respect to x when $x = -1$

Solution

(i) $y = 5x^2 - \frac{5}{x^2} + 2 \Rightarrow y = 5x^2 - 5x^{-2} + 2$

$$\Rightarrow \frac{dy}{dx} = 10x - 5(-2)x^{-3}$$

$$\Rightarrow 10x + \frac{10}{x^3}$$

(ii) At the point $(1, 2)$, $\frac{dy}{dx} = 10(1) + \frac{10}{1} = 20$

(iii) $\frac{dy}{dx}$ is the rate of change of y with respect to x.

Substituting $x = -1$ in the expression for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = 10(-1) + \frac{10}{(-1)^3} = -10 + (-10) = -20$$

Exercise 8A

① Differentiate the following functions.

(i) $y = x^4$ (ii) $y = 2x^3$ (iii) $y = 5x^2$

(iv) $y = 7x^9$ (v) $y = -3x^6$ (vi) $y = 5$

(vii) $y = 10x$ (viii) $y = \frac{1}{4}x^3$ (ix) $y = 2\pi x$

(x) $y = \pi x^2$

② Differentiate the following functions.

(i) $y = 2x^5 + 4x^2$ (ii) $y = 3x^4 + 8x$ (iii) $y = x^3 + 4$

(iv) $y = x - 5x^3$ (v) $y = 4x^3 + 2x$ (vi) $y = 2x + 6$

(vii) $y = 3x^5 + 2$

③ Differentiate the following functions.

(i) $y = 3x^5 + 4x^4 - 3x^2 + 2$

(ii) $y = x^5 + 12x^3 + 3x$

(iii) $y = x^3 + 42x^2 - 5x + 24$

④ Write down the rate of change of the following functions with respect to y.

(i) $y = x^{-4}$ (ii) $y = 3x^{-2}$ (iii) $y = 3x^2 + 4x^{-1}$

(iv) $y = 2x^{-3} - 4$ (v) $y = x^2 + x^{-2}$ (vi) $y = 3x^{-2} + 2x^{-3}$

⑤ Differentiate the following functions.

(i) $y = 3x^2 + \frac{2}{x^3}$ (ii) $y = x^2 + \frac{1}{x^2}$ (iii) $y = 3x^3 + \frac{3}{x^3}$

(iv) $y = \frac{2}{x} - \frac{3}{x^2}$ (v) $y = \frac{1}{2x} - \frac{1}{3x^2}$ (vi) $y = \frac{2}{3x} - \frac{3}{4x^2}$

⑥ A rectangle has length $6x$ and width $3x$.

The area of the rectangle is y.

(i) Write down y in terms of x.

(ii) Work out $\frac{dy}{dx}$.

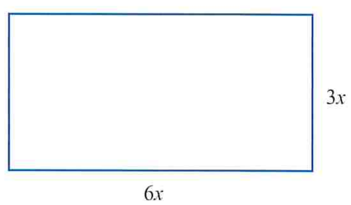


Figure 8.6

7. When a stone is thrown into a lake circular ripples appear centred on the point at which the stone entered the water and spreading outwards. After a time, t seconds, the radius of the circle is r cm where $r = 10t^2$.

   (i) Work out the rate at which the radius is increasing (include the units).

   With time, the definition of the ripples becomes negligible so that after 8 seconds they cannot be seen by the human eye.

   (ii) What is the area of the largest ripple that you can see? Give your answer to the nearest 10 square metres.

8. An expanding sphere has radius $2x$.

   (i) Show that the volume, y, of the sphere is given by the formula $y = \frac{32}{3}\pi x^3$.

   (ii) Work out the rate of change of y with respect to x when $x = 2$

Expressions that first need expanding or dividing

In this case you will need to manipulate the expression into a sum or difference before differentiating.

Example 8.8

Work out $\frac{dy}{dx}$.

(i) $y = x^3(x^2 - 4)$

(ii) $y = \dfrac{x^5 + x^2}{x}$

Solution

(i) Expand to give $y = x^5 - 4x^3$

$$\frac{dy}{dx} = 5x^4 - 12x^2$$

(ii) Make into two fractions $y = \dfrac{x^5}{x} + \dfrac{x^2}{x}$

Cancel to give $y = x^4 + x$

$$\frac{dy}{dx} = 4x^3 + 1$$

Exercise 8B

1. Work out the gradient function for each of the following functions.

   (i) $y = x(x^2 + 2)$ (ii) $y = 2x^2(3x - 4)$

   (iii) $y = (x + 3)(x + 2)$ (iv) $y = (x + 5)(x + 2)$

   (v) $y = x^3(4 + x - x^2)$ (vi) $y = (x + 2)(x - 5)$

2. Work out an expression for the rate of change of y with respect to x for each of the following.

   (i) $y = \dfrac{x^5 + x^3}{4}$ (ii) $y = \dfrac{x^7 + x^3}{x^2}$

   (iii) $y = \dfrac{4x^6 - 2x^2}{x^2}$ (iv) $y = (3x + 1)(x - 2)$

   (v) $y = x^{\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)$ (vi) $y = x^{\frac{1}{2}}\left(x^{\frac{7}{2}} + x^{-\frac{1}{2}}\right)$

③ (i) Simplify $\dfrac{3x^3 - 2x^2}{x}$.

(ii) Use your answer to (i) to differentiate $y = \dfrac{3x^3 - 2x^2}{x}$.

④ Work out the gradient of the curve $y = x^3(x - 2)$ at the point (3, 27).

⑤ Work out the rate of change of y with respect to x for $\dfrac{6x^4 + 2x^5}{2x^3}$ when $x = -1$

⑥ Work out the rate of change of y with respect to x for $y = x^{\frac{1}{3}}\left(x^{\frac{5}{3}} - x^{\frac{2}{3}}\right)$ when $x = -3$

⑦ Work out the gradient of the curve $y = \dfrac{3x^4 + x^2 - 5x}{x}$ at the point (1, −1).

⑧ Work out the gradient of the curve $y = 3\sqrt{x} - \dfrac{3}{\sqrt{x}}$ at the point (4, 4.5).

Note

The rule

$y = x^n \Rightarrow \dfrac{dy}{dx} = nx^{n-1}$

is valid for all values of n but will only be examined when n is an integer.

4 Tangents and normals

Now that you know how to calculate the gradient of a curve at any point you can use this to work out the equation of the tangent at any particular point on the curve.

Example 8.9

(i) Work out the equation of the tangent to the curve $y = 3x^2 - 5x - 2$ at the point (1, −4).

(ii) Sketch the curve and show the tangent on your sketch.

Solution

(i) First work out the gradient function $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = 6x - 5$$

Substitute $x = 1$ into this gradient function to calculate the gradient, m, of the tangent at (1, −4)

$$m = 6 \times 1 - 5$$
$$= 1$$

The equation of the tangent is given by

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = 1(x - 1)$$
$$\Rightarrow \quad y = x - 5$$

$x_1 = 1$, $y_1 = -4$ and $m = 1$

(ii) $y = 3x^2 - 5x - 2$ is a ∪-shaped quadratic curve.

It crosses the x-axis when $3x^2 - 5x - 2 = 0$.

$$\Rightarrow \quad (3x + 1)(x - 2) = 0$$

$$\Rightarrow \quad x = -\frac{1}{3} \quad \text{or} \quad x = 2$$

It crosses the y-axis when $y = -2$

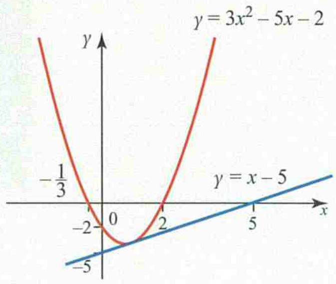


Figure 8.7

The *normal* to a curve at a particular point is the straight line that is at right angles to the tangent at that point (see Figure 8.8). Remember that for perpendicular lines $m_1m_2 = -1$

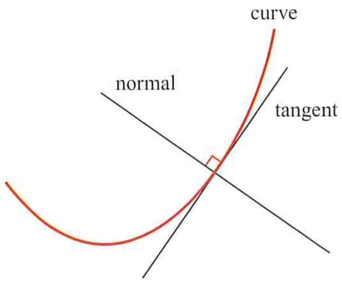


Figure 8.8

Example 8.10

Figure 8.9 is a sketch of the curve $y = x^3 - 3x^2 + 2x$ and the point P(3, 6). Work out the equation of the normal to the curve $y = x^3 - 3x^2 + 2x$ at P.

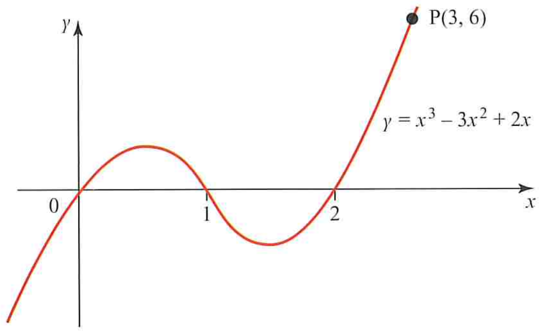


Figure 8.9

Solution

$$y = x^3 - 3x^2 + 2x \Rightarrow \frac{dy}{dx} = 3x^2 - 6x + 2$$

Substitute $x = 3$ to work out the gradient, m_1, of the tangent at the point (3, 6)

$$m_1 = 3 \times (3)^2 - 6 \times 3 + 2 = 11$$

The gradient, m_2, of the normal to the curve at this point is given by

$$m_2 = -\frac{1}{m_1} = -\frac{1}{11}.$$

$m_1m_2 = -1$

The equation of the normal is given by

$$y - y_1 = m_2(x - x_1)$$

(x_1, y_1) is (3, 6).

$$\Rightarrow \quad y - 6 = -\frac{1}{11}(x - 3)$$

Multiply by 11 to eliminate the fraction.

$$\Rightarrow \quad 11y - 66 = -x + 3$$

$$\Rightarrow \quad x + 11y - 69 = 0$$

Example 8.11

Figure 8.10 is a sketch of the curve $y = x^2 + \frac{1}{x}$ for $0 \leqslant x \leqslant 4$ where P is the point (2, 4.5).

(i) Work out the equation of the tangent to the curve at P

(ii) Work out the equation of the normal to the curve at P.

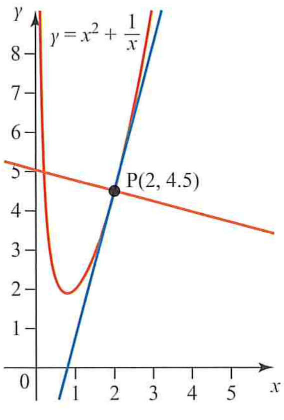


Figure 8.10

Solution

(i) $y = x^2 + \frac{1}{x} = x^2 + x^{-1} \Rightarrow \frac{dy}{dx} = 2x - x^{-2}$

At (2, 4.5) $\frac{dy}{dx} = 4 - \frac{1}{4} = 3.75$ which is the gradient of the tangent.

Using $(y - y_1) = m(x - x_1)$ the equation of the tangent is

$$y - 4.5 = 3.75(x - 2)$$
$$\Rightarrow \quad y - 4.5 = 3.75x - 7.5$$
$$\Rightarrow \quad y = 3.75x - 3$$

(ii) The gradient of the tangent $= 3.75 = \frac{15}{4}$ so the gradient of the normal is $-\frac{4}{15}$.

Using $(y - y_1) = m(x - x_1)$ the equation of the normal is

$$y - 4.5 = -\frac{4}{15}(x - 2)$$
$$\Rightarrow \quad 15(y - 4.5) = -4(x - 2)$$
$$\Rightarrow \quad 15y - 67.5 = -4x + 8$$
$$\Rightarrow \quad 4x + 15y - 75.5 = 0$$

Exercise 8C

① The sketch shows the graph of $y = 5x - x^2$.

The marked point, P, has coordinates (3, 6). Work out

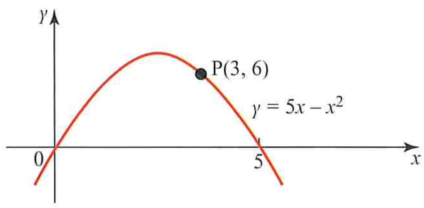


Figure 8.11

(i) the gradient function $\frac{dy}{dx}$

(ii) the gradient of the curve at P

(iii) the equation of the tangent at P

(iv) the equation of the normal at P.

② The sketch shows the graph of $y = 3x^2 - x^3$.

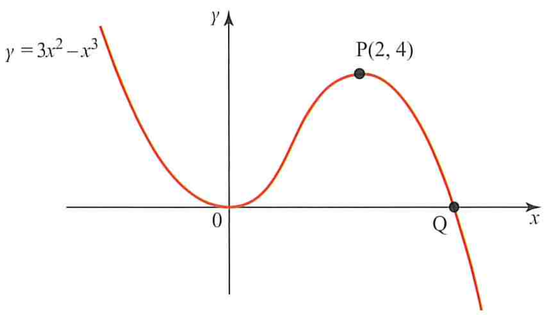


Figure 8.12

(i) The marked point, P, has coordinates (2, 4). Work out

(a) the equation of the tangent at P

(b) the equation of the normal at P.

(ii) The graph touches the x-axis at the origin O and crosses it at the point Q. Work out the equation of the tangent at Q.

(iii) Without further calculation, state the equation of the tangent to the curve at O.

③ The sketch shows the graph of $y = x^5 - x^3$.

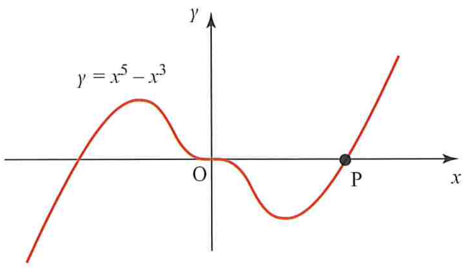


Figure 8.13

(i) Work out the coordinates of the point P where the curve crosses the positive x-axis.

(ii) Work out the equation of the tangent at P.

(iii) Work out the equation of the normal at P.

The tangent at P meets the y-axis at Q and the normal meets the y-axis at R.

(iv) Work out the coordinates of Q and R and hence calculate the area of triangle PQR.

④ (i) Given that $y = x^3 - 3x^2 + 4x + 1$, work out the gradient function $\frac{dy}{dx}$.

(ii) The point P is on the curve $y = x^3 - 3x^2 + 4x + 1$ and its x-coordinate is 2.

(a) Work out the equation of the tangent at P.

(b) Work out the equation of the normal at P.

(iii) Work out the values of x for which the curve has a gradient of 13

⑤ The sketch shows the graph of $y = x^3 - 9x^2 + 23x - 15$

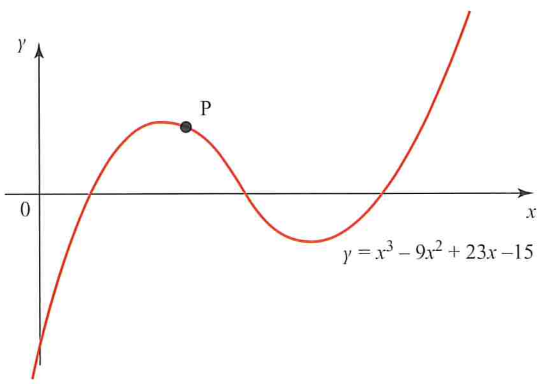


Figure 8.14

The point P marked on the curve has its x-coordinate equal to 2

(i) Work out the equation of the tangent at P.

Q is a point on the curve where the tangent is parallel to the tangent at P.

(ii) Work out the equation of the tangent at Q.

⑥ The point $(2, -8)$ is on the curve $y = x^3 - px + q$.

(i) Identify a relationship between p and q.

The tangent to this curve at the point $(2, -8)$ is parallel to the x-axis.

(ii) Work out the value of p.

(iii) Work out the coordinates of the other point where the tangent is parallel to the x-axis.

(iv) Work out the equation of the normal to the curve at the point where it crosses the y-axis.

⑦ The sketch shows the graph of $y = x^2 - x - 1$

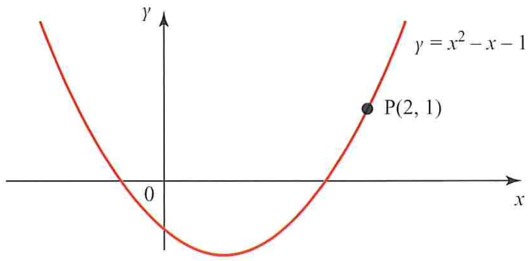


Figure 8.15

(i) Work out the equation of the tangent at the point P.

The normal at a point Q on the curve is parallel to the tangent at P.

(ii) Work out the coordinates of the point Q.

⑧ A curve has the equation $y = (x - 3)(7 - x)$.

(i) Work out the equation of the tangent at the point $(6, 3)$.

(ii) Work out the equation of the normal at the point $(6, 3)$.

(iii) Which one of these lines passes through the origin?

⑨ A curve has the equation $y = 1.5x^3 - 3.5x^2 + 2x$.

(i) Show that the curve passes through the points $(0, 0)$ and $(1, 0)$.

(ii) Work out the equations of the tangents and normals at each of these points.

(iii) What shape is formed by the four lines in part (ii)?

⑩ Figure 8.16 shows the curve with the equation $y = x^2 + \frac{2}{x}$ for $x > 0$

(i) Work out the gradient function $\frac{dy}{dx}$ and calculate the coordinates of the minimum point.

(ii) State the equations of the tangent and the normal at that minimum point.

(iii) Work out the equations of the tangent and normal at the point where $x = 2$

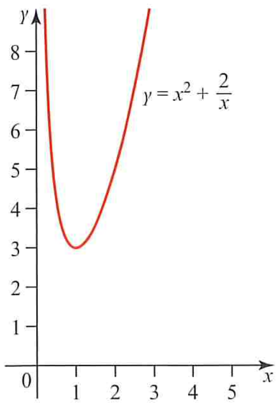


Figure 8.16

Chapter 8 Calculus

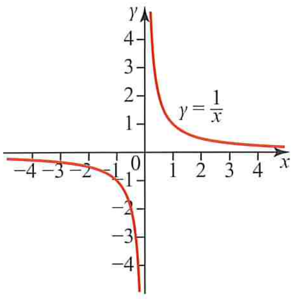


Figure 8.17

Discussion point

➜ The curve on the left has equation $y = \frac{1}{x}$ for $x \neq 0$

(i) Work out the equations of the tangents at the points $(-1, -1)$ and $(1, 1)$.

(ii) What do you notice about these lines?

(iii) Work out the equations of the normals at the points $(-1, -1)$ and $(1, 1)$.

(iv) What do you notice about these?

5 Increasing and decreasing functions

A function $y = \mathrm{f}(x)$ is

- increasing if $\frac{dy}{dx} > 0$
- decreasing if $\frac{dy}{dx} < 0$

Some functions are increasing or decreasing over their whole domain.

For example, $y = 3 - 2x$ is a decreasing function for all real values of x because $\frac{dy}{dx} = -2$ which is < 0

Other functions are increasing over parts of their domain and decreasing over others.

Example 8.12

Work out the values of x for which the function $y = x^2 - 4x + 1$ is an increasing function.

Solution

First work out $\frac{dy}{dx}$ $\qquad \frac{dy}{dx} = 2x - 4$

To be an increasing function $\frac{dy}{dx} > 0$ $\qquad 2x - 4 > 0$

$$2x > 4$$

$$x > 2$$

Exercise 8D

① Work out the values of x for which the following functions are increasing.

(i) $y = x^2 + 4$
(ii) $y = 2x - 3$
(iii) $y = x^2 + 2x - 5$
(iv) $y = x^2 - 3x$
(v) $y = 3x^2 + 4x + 7$
(vi) $y = (x + 6)(x - 2)$
(vii) $y = x^3 - 2x^2$
(viii) $y = x^3 + 6x^2 - 15x$
(ix) $y = x^3 - 3x^2 - 9x + 1$

② Work out the values of x for which the following functions are decreasing.

(i) $y = 4x^2$
(ii) $y = x^2 - 6x + 2$
(iii) $y = x(x + 2)$
(iv) $y = 3 + 4x - x^2$
(v) $y = 12 - x$
(vi) $y = (2x + 1)^2$
(vii) $y = \frac{1}{3}x^3 + x^2$
(viii) $y = 2x^3 - 3x^2 - 72x$
(ix) $y = 27x - x^3$

③ Prove that $y = \frac{1}{3}x^3 + 2x^2 + 7x + 1$ is an increasing function for all values of x.

④ Prove that $y = x^3 - 6x^2 + 27x - 4$ is an increasing function for all values of x.

⑤ Work out the values of x for which $y = x^2 + \frac{2}{x}$ is an increasing function.

⑥ Prove that $y = 12 - 2x - x^3$ is a decreasing function for all values of x.

⑦ Prove that $y = \frac{1}{x}$ is a decreasing function for all $x \neq 0$

⑧ Work out the values of x for which the following functions are

(a) increasing (b) decreasing.

(i) $y = x + \frac{1}{x}$ (ii) $y = x - \frac{1}{x}$

(iii) $y = x^2 + \frac{1}{x^2}$ (iv) $y = x^2 - \frac{1}{x^2}$

⑨ Air is being pumped into a spherical balloon at the rate of $1000\,\text{cm}^3\,\text{s}^{-1}$. Initially the balloon contains no air. (The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$).

(i) Calculate the volume V of the balloon after 10 seconds.

(ii) Calculate the volume of the balloon after t seconds.

(iii) State the value of $\frac{dV}{dt}$.

(iv) Calculate the radius of the balloon after t seconds.

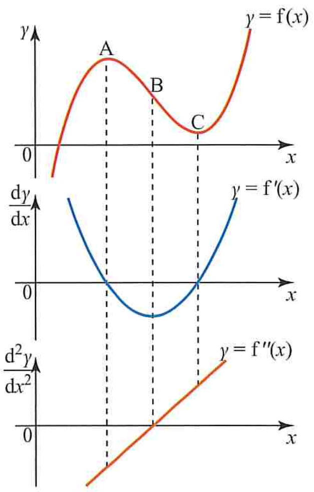


Figure 8.18

6 The second derivative

Figure 8.18 shows a sketch of a function $y = f(x)$ with a sketch of the corresponding gradient function, $\frac{dy}{dx} = f'(x)$ below it.

The third graph shows the gradient of the function $y = f'(x)$, denoted by $y = f''(x)$.

The gradient of any point on the curve of $\frac{dy}{dx}$ is found by differentiating $\frac{dy}{dx}$ and is given by $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. This is written as $\frac{d^2y}{dx^2}$ or $y = f''(x)$ and is called the second derivative.

! $\frac{d^2y}{dx^2}$ is not the same as $\left(\frac{dy}{dx}\right)^2$.

Example 8.13

Given that $y = 2x^3 - 4x^2 + 3x - 1$, work out $\frac{d^2y}{dx^2}$.

Solution

$$\frac{dy}{dx} = 6x^2 - 8x + 3$$

$$\frac{d^2y}{dx^2} = 12x - 8$$

Example 8.14

A ball is thrown upwards with a speed of $20\,\mathrm{m\,s^{-1}}$. Its height h m above the ground after a time of t seconds is given by $h = 1 + 20t - 5t^2$.

(i) Work out $\frac{dh}{dt}$ and say what this represents.

(ii) Calculate the maximum height reached by the ball and the time at which this height is reached.

(iii) Work out the rate of change of $\frac{dh}{dt}$, written as $\frac{d^2h}{dt^2}$, and say what this represents.

(iv) Sketch the graph of h against t.

Solution

(i) $\frac{dh}{dt} = 20 - 10t$

This represents the velocity of the stone.

When the velocity is positive the stone is moving upwards and when it is negative it is moving downwards. When it is zero it is stationary at the highest point.

(ii) The maximum height is reached when the ball is instantaneously at rest. This means that $\frac{dh}{dt} = 0$ giving $20 - 10t = 0$, so $t = 2$

When $t = 2$, $h = 1 + 20(2) - 5(4) = 21$

The maximum height is 21 m above the ground after a time of 2 seconds.

(iii) $\frac{dh}{dt} = 20 - 10t \quad \Rightarrow \quad \frac{d^2h}{dt^2} = -10$

The rate of change of velocity is acceleration, and the positive direction is measured upwards, so this means that the acceleration of the ball is $-10\,\mathrm{m\,s^{-2}}$ upwards, which is the same as saying that the ball is decelerating, i.e. slowing down, at a rate of $10\,\mathrm{m\,s^{-2}}$ as it travels upwards. On its descent it will accelerate at a rate of $10\,\mathrm{m\,s^{-2}}$ downwards.

(iv) $h = 1 + 20t - 5t^2$ is represented by a quadratic graph passing through (0, 1) and having a maximum point at (2, 21).

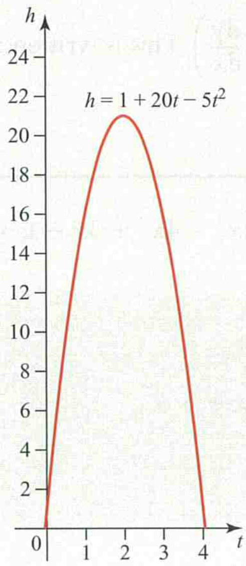


Figure 8.19

Exercise 8E

① Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = 3x^3 + 3x$ (ii) $y = x^5 - 25$

(iii) $y = 3x - 5x^4$

② Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = x^4 - 2x^2 + 5x - 4$ (ii) $y = 2x^3 + 3x - 4$

(iii) $y = x^3 - 2x^2 + 1$

③ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions. Remember that when an expression involves brackets you need to multiply out before differentiating.

(i) $y = (2x - 1)(x + 2)$ (ii) $y = (2x - 1)^2$

(iii) $y = (1 - 3x)(2x - 3)$

④ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = 3(x - 2)\left(x^2 - 2x + 3\right)$ (ii) $y = 2x^2(x - 1)^2$

(iii) $y = x^3(3x + 1)^2$

⑤ The sum of two numbers x and y is 13 and their product P is 40.

(i) Write down an expression for y in terms of x.

(ii) Write down an expression for P in terms of x.

(iii) Write down expressions for $\frac{dy}{dx}$ and $\frac{dP}{dx}$.

(iv) Write down the rate of change of $\frac{dP}{dx}$.

⑥ For the curve $y = 3x^3 - 2x^2 - 6x - 4$

(i) write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(ii) work out the gradient of the curve at the points $(-1, -7)$, $(1, -9)$ and $(2, 0)$.

(iii) work out the rate of change of the gradient at each of these points.

⑦ A formula which you will meet in Mechanics or Physics is $s = ut + \frac{1}{2}at^2$, where the letters in this case are t = time, u is the initial velocity (which will be a constant, or zero if starting from rest), a is the acceleration (which must also be constant, for this formula) and s is the distance travelled. The only variables in the formula are s and t. Using this formula $\frac{ds}{dt}$ will give the velocity after a time t has elapsed.

(i) Work out $\frac{ds}{dt}$ and hence the velocity after 12 seconds when the distance is measured in metres and time in seconds.

(ii) Work out $\frac{d^2s}{dt^2}$.

7 Stationary points

ACTIVITY 8.5

(i) Plot the graph of $y = x^4 - 3x^3 - x^2 + 3x$, taking values of x from -1.5 to $+3.5$ in steps of 0.5

You will need your y-axis to go from -10 to $+20$

Alternatively, if you have access to a graphics calculator or graphing software you could use that.

(ii) Describe the curve as x goes from -1.5 to 3.5

A *stationary point* on a curve is one where the gradient is zero. This means that the tangents to the curve at these points are horizontal. Figure 8.20 shows a curve with two stationary points A and B.

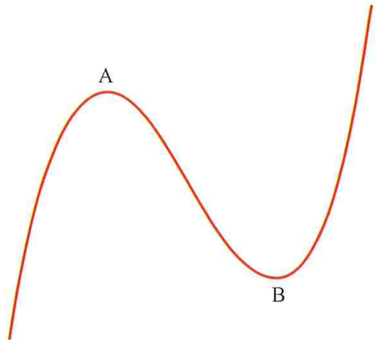


Figure 8.20

Note

Maximum and minimum points are turning points. Questions on this specification will not use the term 'turning points'.

As the curve passes through the points A and B it changes direction completely. At A the gradient changes from positive to negative and at B from negative to positive. A is called a *maximum* point and B is a *minimum* point.

ACTIVITY 8.6

Figure 8.21 shows the graph of $y = \cos x$.
Describe the gradient of the curve, using the words 'positive', 'negative', 'zero', 'increasing' and 'decreasing', as x increases from 0° to 360°.

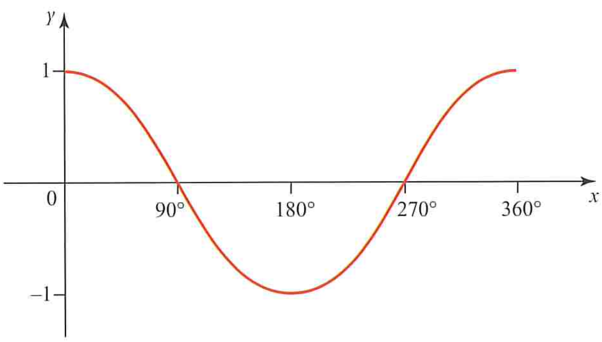


Figure 8.21

Maximum and minimum points

Figure 8.22 shows the graph of $y = 4x - x^2$. It has a maximum point at (2, 4).

You can see that

- at the maximum point the gradient $\frac{dy}{dx}$ is zero
- the gradient is positive to the left of the maximum and negative to the right of it.

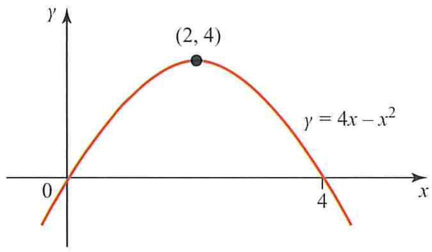


Figure 8.22

This is true for any maximum point (see Figure 8.23).

In the same way, for any minimum point (see Figure 8.24)

- the gradient is zero at the minimum
- the gradient goes from negative to zero to positive.

You can see that the gradient function is decreasing (+, 0, –) through a maximum point and increasing (–, 0, +) through a minimum point.

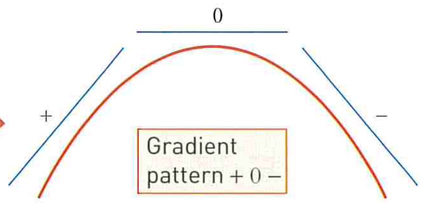


Figure 8.23

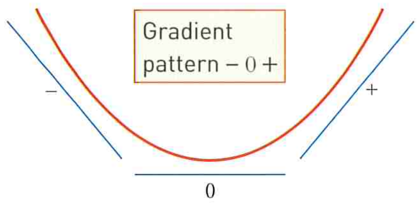


Figure 8.24

Once you have found the position and type of any stationary points, you can use this information to sketch the curve.

Example 8.15

For the curve $y = x^3 - 12x + 3$

(i) work out $\frac{dy}{dx}$ and the values of x for which $\frac{dy}{dx} = 0$

(ii) classify the points on the curve with these x-values

(iii) work out the corresponding y-values

(iv) sketch the curve.

Solution

(i) $\frac{dy}{dx} = 3x^2 - 12$

When $\frac{dy}{dx} = 0$

$$3x^2 - 12 = 0$$
$$\Rightarrow \quad 3(x^2 - 4) = 0$$
$$\Rightarrow \quad 3(x + 2)(x - 2) = 0$$
$$\Rightarrow \quad x = -2 \quad \text{or} \quad x = 2$$

(ii) For $x = -2$

$$x = -3 \quad \Rightarrow \quad \frac{dy}{dx} = 3(-3)^2 - 12 = +15$$

$$x = -1 \quad \Rightarrow \quad \frac{dy}{dx} = 3(-1)^2 - 12 = -9$$

Gradient pattern + 0 −

$\Rightarrow$ maximum point when $x = -2$

For $x = +2$

$$x = 1 \quad \Rightarrow \quad \frac{dy}{dx} = 3(1)^2 - 12 = -9$$

$$x = 3 \quad \Rightarrow \quad \frac{dy}{dx} = 3(3)^2 - 12 = +15$$

Gradient pattern − 0 +

$\Rightarrow$ minimum point when $x = +2$

(iii) When $x = -2$, $y = (-2)^3 - 12(-2) + 3 = 19$

When $x = +2$, $y = (2)^3 - 12(2) + 3 = -13$

(iv) There is a maximum at $(-2, 19)$ and a minimum at $(2, -13)$.

The only other information you need to sketch the curve is the value of y when $x = 0$. This tells you where the curve crosses the y-axis.

When $x = 0$, $y = (0)^3 - 12(0) + 3 = 3$

The graph of $y = x^3 - 12x + 3$ is shown in Figure 8.25.

Discussion point

➜ Why can you be confident about continuing the sketch of the curve beyond the x-values of the stationary points?

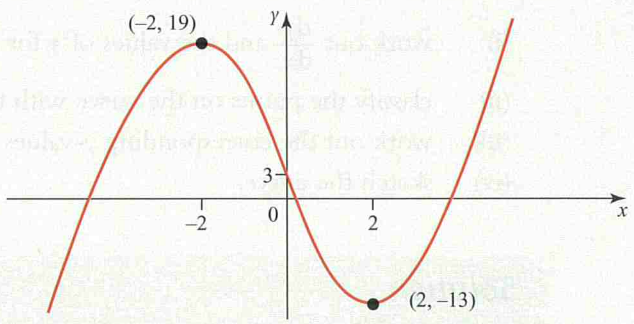


Figure 8.25

Discussion point

➜ In Example 8.15 you did not work out the coordinates of the points where the curve crosses the x-axis.

(i) Why was this?

(ii) Under what circumstances would you work out these points?

Example 8.16

Identify all the stationary points on the curve of $y = x^4 - 2x^3 + x^2 - 2$ and sketch the curve.

Solution

$$\frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 2x(2x^2 - 3x + 1) = 0$$

$$\Rightarrow \quad 2x(2x - 1)(x - 1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = 0.5 \text{ or } x = 1$$

You may find it helpful to summarise your working in a table. You can find the various signs, + or –, by taking a test point in each interval, for example, $x = 0.25$ in the interval $0 < x < 0.5$

	$x < 0$	0	$0 < x < 0.5$	0.5	$0.5 < x < 1$	1	$x > 1$
sign of $\frac{dy}{dx}$	–	0	+	0	–	0	+
stationary point		min		max		min	

When $x = 0$: $y = (0)^4 - 2(0)^3 + (0)^2 - 2 = -2$

When $x = 0.5$: $y = (0.5)^4 - 2(0.5)^3 + (0.5)^2 - 2 = -1.9375$

When $x = 1$: $y = (1)^4 - 2(1)^3 + (1)^2 - 2 = -2$

Therefore $(0.5, -1.9375)$ is a maximum stationary point and $(0, -2)$ and $(1, -2)$ are both minimum stationary points.

The graph of $y = x^4 - 2x^3 + x^2 - 2$ is shown in Figure 8.26.

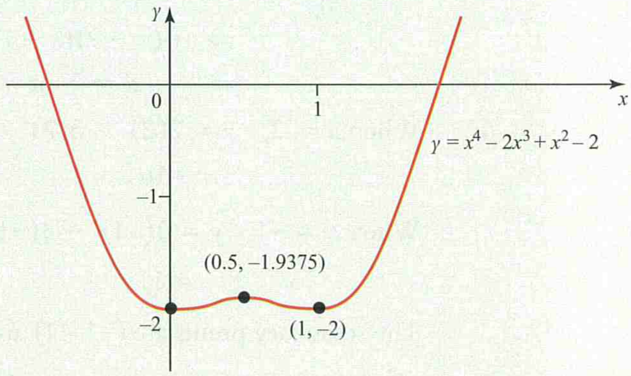


Figure 8.26

The method above, setting out a table of values, is rather tedious but there is an alternative way of identifying whether a stationary point is a maximum or a minimum using the second derivative. Recall that $\frac{d^2y}{dx^2}$ represents the rate of change of $\frac{dy}{dx}$, i.e. the rate of change of the gradient.

Stationary points

Figures 8.23 and 8.24 illustrated that at a minimum/maximum the gradient function is increasing/decreasing. At a point where $\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} > 0$ gives a minimum

$\frac{d^2y}{dx^2} < 0$ gives a maximum

If $\frac{d^2y}{dx^2}$ is positive at a stationary point (i.e. where $\frac{dy}{dx} = 0$), then the gradient must go from negative to positive, in which case the turning point will be a minimum.

Conversely, if $\frac{d^2y}{dx^2}$ is negative at a stationary point, then the gradient must go from positive to negative which will indicate a maximum turning point.

Note

If $\frac{d^2y}{dx^2} = 0$ at the stationary point, it is not possible to use this method and you will have to go back to the method of checking the gradient on each side of the stationary point.

Example 8.17

Given that $y = 2x^3 - 3x^2 - 12x + 4$

(i) work out $\frac{dy}{dx}$, and the values where $\frac{dy}{dx} = 0$

(ii) work out the coordinates of each of the stationary points

(iii) work out the value of $\frac{d^2y}{dx^2}$ at each of the stationary points and hence determine the nature of each one

(iv) sketch the graph of $y = 2x^3 - 3x^2 - 12x + 4$

Solution

(i) $\frac{dy}{dx} = 6x^2 - 6x - 12$

When $\frac{dy}{dx} = 0$, $\quad 6(x^2 - x - 2) = 0$

$\Rightarrow \quad 6(x - 2)(x + 1) = 0$

$\Rightarrow \quad x = 2 \text{ or } x = -1$

(ii) When $x = 2$, $y = 2(2)^3 - 3(2)^2 - 12(2) + 4$

$= -16$

When $x = -1$, $y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 4$

$= 11$

The stationary points are $(-1, 11)$ and $(2, -16)$

(iii) $\frac{d^2y}{dx^2} = 12x - 6$

When $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$ so $(-1, 11)$ is a maximum point.

When $x = 2$, $\frac{d^2y}{dx^2} = +18 > 0$ so $(2, -16)$ is a minimum point.

(iv) The curve crosses the y-axis when $x = 0$, i.e. the point (0, 4). This information, together with the positions of the stationary points, is sufficient to enable you to sketch the curve.

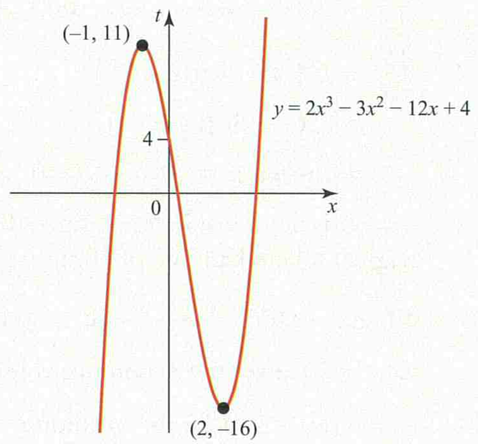


Figure 8.27

Applications of maxima and minima

Example 8.18

An open box is made from a square sheet of cardboard, with sides 60 cm long, by cutting out a square from each corner, folding up the sides and joining the cut edges.

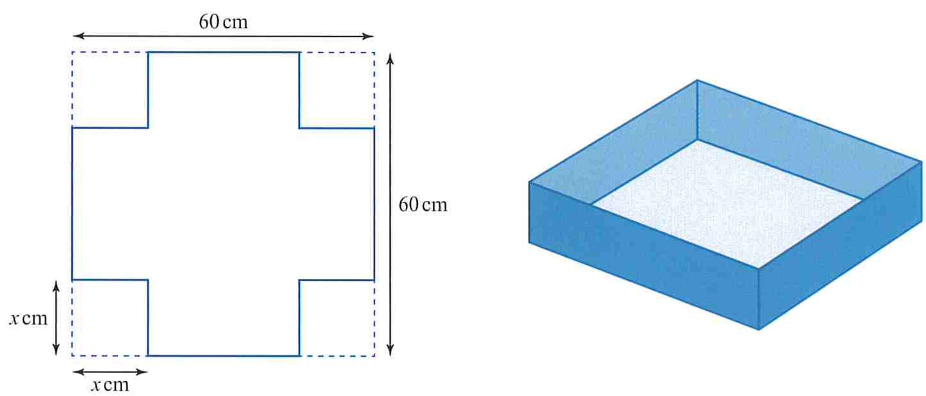


Figure 8.28

(i) Write down an expression for the volume V of the box in terms of x.

(ii) Work out $\frac{dV}{dx}$ and $\frac{d^2V}{dx^2}$ and use these to calculate the value of x giving the maximum volume.

(iii) What is the maximum volume?

Solution

(i) Volume = area of base × depth

$$= (60 - 2x)^2 \times x$$

$$= x(60 - 2x)^2 \text{ cm}^3$$

(ii) Expanding this, $V = x(3600 - 240x + 4x^2)$

$$= 3600x - 240x^2 + 4x^3$$

$$\frac{dV}{dx} = 3600 - 480x + 12x^2 \text{ and } \frac{d^2V}{dx^2} = -480 + 24x$$

$$\frac{dV}{dx} = 12(300 - 40x + x^2)$$

$$= 12(x - 30)(x - 10)$$

$$= 0 \text{ when } x = 10 \text{ or } x = 30$$

$x = 30$ is not a viable result, since there would be no box (the original square had side of 60 cm) so this value must be rejected.

When $x = 100$, $\frac{d^2V}{dx^2} = -480 + 24(10) = -240$ (i.e. negative)

$\Rightarrow x = 10$ gives the maximum volume.

(iii) $V = x(60 - 2x)^2$ so the maximum volume is 10×40^2 cm^3

Maximum volume is 16 000 cm^3.

Example 8.19

The perimeter of a rectangle has a fixed length. Show that the area of the rectangle is greatest when the rectangle is a square.

Solution

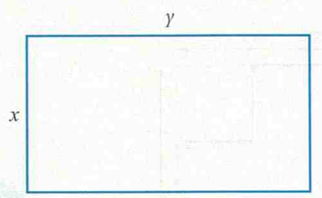


Figure 8.29

The perimeter $2x + 2y = c$, where c is a constant.

$$\Rightarrow 2y = c - 2x$$

$$\Rightarrow y = \frac{1}{2}c - x$$

Area $A = xy = x\left(\frac{1}{2}c - x\right)$

$$= \frac{1}{2}cx - x^2$$

Differentiating: $\frac{dA}{dx} = \frac{1}{2}c - 2x$

$$= 0 \text{ when } x = \frac{c}{4}$$

$\frac{d^2A}{dx^2} = -2 \Rightarrow$ stationary point is a maximum.

Since $y = \frac{1}{2}c - x$, when $x = \frac{c}{4}$, then $y = \frac{c}{4}$ also, so the shape is a square.

Consequently the area is greatest when the rectangle is a square.

Example 8.20

A closed cardboard box with a square base is to be constructed to have a capacity of 216 000 cm³. What dimensions use a minimum amount of cardboard?

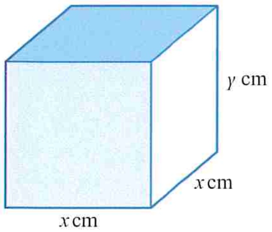


Figure 8.30

Solution

Area of base = area of top = x^2.

Area of sides = $4 \times xy$.

The area of cardboard used $A = 2x^2 + 4xy$.

Volume of the box $V = x^2y \Rightarrow 216\,000 = x^2y$

$$\Rightarrow y = \frac{216\,000}{x^2} = 216\,000x^{-2}.$$

Substituting for y in the expression for A gives

$$A = 2x^2 + 4x \times 216\,000x^{-2}$$
$$= 2x^2 + 864\,000x^{-1}.$$

Differentiating: $\dfrac{dA}{dx} = 4x - 864\,000x^{-2}$

$= 0$ when $4x = 864\,000x^{-2}$

$\Rightarrow x^3 = 216\,000$

$\Rightarrow \quad x = 60$

Differentiating again: $\dfrac{d^2A}{dx^2} = 4 + 1728\,000x^{-3}$

> 0 when $x = 60$ so a minimum.

When $x = 60$, $y = \dfrac{216\,000}{60^2} = 60$

The dimensions of the box using the minimum amount of cardboard are 60 cm × 60 cm × 60 cm.

Example 8.21

$P = 32x^2 + \dfrac{8}{x}$.

Show that P has a minimum value when $x = \dfrac{1}{2}$ and work out the minimum value of P.

Solution

$$P = 32x^2 + \frac{8}{x}$$
$$= 32x^2 + 8x^{-1}$$

$$\frac{dP}{dx} = 64x - 8x^{-2}$$

$= 0$ when $64x = 8x^{-2}$

$$\Rightarrow \quad x^3 = \frac{8}{64} = \frac{1}{8}$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$\frac{d^2P}{dx^2} = 64 + 16x^{-3}$$

$$x = \frac{1}{2} \quad \Rightarrow \quad \frac{d^2P}{dx^2} = 192 \text{ (i.e. positive)}$$

P is a minimum when $x = \dfrac{1}{2}$

The minimum value of P is $32 \times 0.5^2 + \dfrac{8}{0.5} = 24$

Exercise 8F

If you have access to a graphic calculator you will find it helpful to use it to check your answers.

① For each of the curves given below

(a) work out $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$

(b) work out the value(s) of $\frac{d^2y}{dx^2}$ at those points

(c) classify the point(s) on the curve with these x-values

(d) work out the corresponding y-value(s)

(e) sketch the curve.

(i) $y = 1 + x - 2x^2$

(ii) $y = 12x + 3x^2 - 2x^3$

(iii) $y = x^3 - 4x^2 + 9$

(iv) $y = x(x - 1)^2$

(v) $y = x^2(x - 1)^2$

(vi) $y = x^3 - 48x$

(vii) $y = x^3 + 6x^2 - 36x + 25$

(viii) $y = 2x^3 - 15x^2 + 24x + 8$

② The graph of $y = px + qx^2$ passes through the point $(3, -15)$ and its gradient at that point is -14

(i) Work out the values of p and q.

(ii) Calculate the maximum value of y and state the value of x at which it occurs.

③ (i) Identify the stationary points of the function $f(x) = x^2(3x^2 - 2x - 3)$ and distinguish between them.

(ii) Sketch the curve $y = f(x)$.

④ The curve $y = ax^2 + bx + c$ crosses the y-axis at the point $(0, 2)$ and has a minimum point at $(3, 1)$.

(i) Work out the equation of the curve.

(ii) Check that the stationary point is a minimum.

⑤ The sum of two positive numbers p and q is 12

(i) Write q in terms of p.

(ii) S is the sum of the squares of the two numbers. Write down an expression for S in terms of p.

(iii) Work out the least value of S, checking that it is a minimum.

⑥ The sum of two positive numbers a and b is 40

(i) Write $2ab$ in terms of a.

(ii) Work out values of a and b when $2ab$ is a maximum, checking that it is a maximum.

(iii) Work out the maximum value of $2ab$.

⑦ x and y are two positive numbers whose sum is 10

(i) Express $P = xy^2$ in terms of x.

(ii) Work out the values of x for which $\frac{dP}{dx} = 0$

(iii) Use the second derivative test to identify which one gives the maximum value of P.

(iv) Comment on the implication of the other value of x that you calculated.

⑧ Netty and Mackenzie are going to climb a mountain and the equation of their path is given by $10y = x + 4x^2 - x^3$ for $x \geqslant 0$. Distances horizontally and vertically are measured in units of 1000 metres. Give all answers to 3 significant figures.

(i) How far away, horizontally, is the summit?

(ii) How much higher is the summit than where they are now?

⑨ The base of a cuboid is x cm by x cm and its height is y cm. Its volume is $216\,\text{cm}^3$.

(i) Write down an expression for the surface area in terms of x.

(ii) Work out the dimensions that give the minimum surface area, proving that this is a minimum.

FUTURE USES

- This work will be extended if you study Mathematics at a higher level.
- At A-Level you will learn additional formulae to deal with more complex algebraic products and quotients.
- You will also use integration to calculate the area under a curve or between two curves and to calculate the volume when a curve is rotated about the x- or y-axes.
- There are also applications in other subjects, for example, Kinematics, Physics and Economics.

REAL-WORLD CONTEXT

Differentiation is used in the study of motion.
It is also the basis of differential equations which can be used to solve problems involving growth and decay.
Navier-Stokes equations, which are a particular form of differential equation, are vital to video-gaming and also help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution and many other things.

LEARNING OUTCOMES

Now you have finished the chapter, you should be able to

- differentiate positive and negative powers of a variable such as x
- differentiate sums and differences of functions of x
- differentiate functions of x that first need expanding or dividing
- use differentiation to work out the gradient of a curve
- use this information to identify stationary points on a polynomial curve
- derive the equation of a tangent or a normal to a curve
- identify when a function is increasing and when it is decreasing
- calculate the position of any stationary points on the curve
- use the second derivative to determine the nature of any stationary points.

KEY POINTS

1 $y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}$

$y = c \quad \Rightarrow \quad \frac{dy}{dx} = 0$

where n is a positive integer and k and c are constants.

2 $y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x)$.

3 For the tangent and normal at (x_1, y_1)

- the gradient of the tangent, m_1 = the value of $\frac{dy}{dx}$
- the gradient of the normal, $m_2 = -\frac{1}{m_1}$
- the equation of the tangent is $y - y_1 = m_1(x - x_1)$
- the equation of the normal is $y - y_1 = m_2(x - x_1)$.

4 A function $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$

A function $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$

5 The second derivative is obtained by differentiating $\frac{dy}{dx}$ and is denoted by $\frac{d^2y}{dx^2}$.

6 At a stationary point, $\frac{dy}{dx} = 0$

The nature of the stationary point can be determined by looking at the sign of the gradient just either side of it.

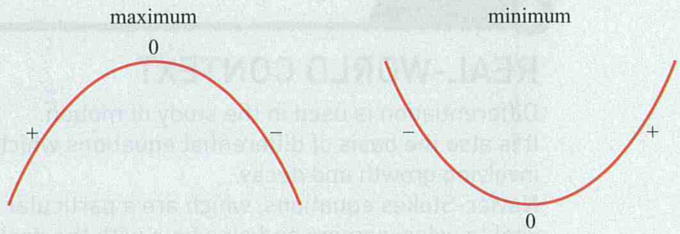


Figure 8.31

7 The sign of $\frac{d^2y}{dx^2}$ is an alternative way of determining the nature of the stationary point:

- If $\frac{d^2y}{dx^2} < 0$ at the stationary point, the point is a maximum.
- If $\frac{d^2y}{dx^2} > 0$ at the stationary point, the point is a minimum.
- If $\frac{d^2y}{dx^2} = 0$ at the stationary point, the result is inconclusive and you need to check the values of $\frac{dy}{dx}$ as in point 6.

9 Matrices

Nothing tends so much to the advancement of knowledge as the application of a new instrument.

Sir Humphry Davy

Prior knowledge

- Students should be familiar with enlargements, reflections and rotations which were met when studying transformations at GCSE.
- It is helpful to be familiar with vectors which are used to describe translations in GCSE.
- Some problem-style questions require students to be comfortable solving equations, including quadratic and simultaneous equations – see Chapters 1, 2 and 4.

An arrangement of information presented in columns and rows is called a matrix.

As an example, the number of male students and female students in two tutor groups are shown in the following matrix.

$$\begin{array}{r} \\ \text{Tutor group A} \\ \text{Tutor group B} \end{array} \begin{array}{c} \quad \text{Male} \quad \text{Female} \\ \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} \end{array}$$

Each of the four numbers in the matrix is called an **element**.

The matrix on the previous page has 2 rows and 2 columns and so is referred to as a 2 × 2 (read as 'two by two') matrix.

$\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ has 2 rows and 1 column. It is an example of a 2 × 1 matrix.

1 Multiplying matrices

Multiplication of a matrix by a scalar

In this context, a scalar is a number.

Each element of the matrix is multiplied by the scalar.

Example 9.1

Given that $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$, what is 4**A**?

Solution

$$4\begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & -12 \\ -4 & 20 \end{bmatrix}$$

Multiplication of a 2 × 2 matrix by a 2 × 1 matrix

Each row of the 2 × 2 matrix is multiplied by the column of the 2 × 1 matrix.

Example 9.2

Given that $\mathbf{P} = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, calculate the product **PQ**. ← $\mathbf{PQ} = \mathbf{P} \times \mathbf{Q}$

Solution

$$\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} (-3) \times 6 + (-1) \times 5 \\ 4 \times 6 + 2 \times 5 \end{bmatrix}$$
$$= \begin{bmatrix} -23 \\ 34 \end{bmatrix}$$

Matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second matrix. So the product

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ cannot be calculated.

Multiplication of a 2 × 2 matrix by a 2 × 2 matrix

Each row of the first 2 × 2 matrix is multiplied by each column of the second 2 × 2 matrix.

Example 9.3

Given that $\mathbf{C} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$, calculate the product **CD**.

Solution

$$\begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 \times 2 + (-1) \times (-3) & 0 \times 4 + (-1) \times 1 \\ (-2) \times 2 + 3 \times (-3) & (-2) \times 4 + 3 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ -13 & -5 \end{bmatrix}$$

Equating elements

If two matrices are equal, then their corresponding elements can be equated.

This principle is used in the following example.

Example 9.4

Given that $\begin{bmatrix} 3 & a \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -3 & b \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ -7 & 0 \end{bmatrix}$

work out the values of a and b.

Solution

Multiplying out the left-hand side gives $\begin{bmatrix} 6-3a & -6+ab \\ -7 & 4+b \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ -7 & 0 \end{bmatrix}$

Equating elements in row 1 column 1 gives

$$6 - 3a = 12$$
$$6 - 12 = 3a$$
$$-6 = 3a$$
$$a = -2$$

Equating elements in row 2 column 2 gives

$$4 + b = 0$$
$$b = -4$$

Check by comparing the elements in row 1 column 2:
$-6 + ab = -6 + (-2) \times (-4) = 2$

Discussion point

➜ Can you identify matrices $\mathbf{P}$ and $\mathbf{Q}$ for which $\mathbf{PQ} = \mathbf{QP}$?

ACTIVITY 9.1

$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

Calculate the products **AB** and **BA**. What do you notice?

Exercise 9A

① $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 6 & -2 \\ -3 & -1 \end{bmatrix}$ $\mathbf{D} = \begin{bmatrix} 0 & 0 \\ -3 & -5 \end{bmatrix}$

$\mathbf{E} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\mathbf{F} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$\mathbf{G} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\mathbf{H} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Work out

(i) 4**A** (ii) 2**D** (iii) **AF**
(iv) **CE** (v) **DH** (vi) **BH**
(vii) **AB** (viii) **BA** (ix) **BC**
(x) **CB** (xi) **DA** (xii) **BD**
(xiii) **AC** (xiv) **DC**

② Work out the value of p in each of the following.

(i) $\begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & -1 \\ p & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

(iii) $\begin{bmatrix} p & 1 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ p \end{bmatrix} = \begin{bmatrix} 2 \\ 17 \end{bmatrix}$

(iv) $\begin{bmatrix} p & 4p \\ p & -2p \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ p & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 16 & 9 \end{bmatrix}$

(vi) $\begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2p \\ -1 & p \end{bmatrix} = \begin{bmatrix} 9 & -14 \\ 0 & 0 \end{bmatrix}$

③ Work out the values of x and y in each of the following.

(i) $\begin{bmatrix} 2 & 1 \\ 1 & y \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & x \\ 2y & 3y \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$

(iii) $\begin{bmatrix} -3 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ x & y \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ -4 & 10 \end{bmatrix}$

(iv) $\begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ x & y \end{bmatrix} = \begin{bmatrix} -9 & -5 \\ -2 & -4 \end{bmatrix}$

④ Given that $\begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

(i) write down two equations in x and y

(ii) work out x and y by solving the pair of simultaneous equations.

⑤ Work out the values of a and b in each of the following.

(i) $\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 21 \end{bmatrix}$

(ii) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = 6\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & b \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & 2a \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 11 \\ 2 & -2 \end{bmatrix}$

(iv) $\begin{bmatrix} 5 & 1 \\ 3a & b+1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 3 \end{bmatrix}$

PS ⑥ Given that $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, work out the values of a, b, c and d.

PS ⑦ $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 0 & k \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$. Work out the value of k such that $\mathbf{AB} = \mathbf{BA}$.

PS ⑧ Given that $\begin{bmatrix} p & 4 \\ 1 & q \end{bmatrix}\begin{bmatrix} r & p-4 \\ q+8 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 3p+4 \\ -17 & -3p \end{bmatrix}$, work out the values of p, q and r.

PS ⑨ (i) Calculate the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(ii) Hence write down the matrix $\mathbf{M}$, in terms of a, b, c and d, such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Discussion point

Are there any values of a, b, c and d, for which the matrix $\mathbf{M}$ in question 9 does not exist?

2 Transformations

If a point P(x, y) is transformed to point P′(x', y'), then P is said to have been mapped to the **image** P′.

For example,

(i) when the point (4, 2) is transformed by reflection in the y-axis, the image point is (−4, 2). So (4, 2) has been mapped to (−4, 2).

(ii) when the point (3, −1) is transformed by rotation through 270°, centre O, the image point is (−1, −3). So (3, −1) has been mapped to (−1, −3).

Note

Assume that a rotation is anticlockwise unless told otherwise.

Matrix transformations

A transformation can be defined by a matrix.

For a transformation that maps $(1, 0)$ to (a, c), and $(0, 1)$ to (b, d)

the transformation matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

When such a matrix acts on a point, we use the point's position vector.

The position vector of any point is measured from the origin, and is the point's coordinates in vertical form with the x-coordinate on top.

So the position vector of $(-1, 5)$ is $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

When a transformation acts on a point, multiply its matrix by the point's position vector. The product is the position vector of the image.

For a transformation matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acting on the point (x, y), the image (x', y') is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

Example 9.5

Work out the image of point $(-2, 5)$ for the transformation defined by matrix $\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$$

The image point is $(11, 9)$.

ACTIVITY 9.2

(i) For the transformation defined by matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, work out the image of each of the following points.

$(3, 2)$ $(-1, 5)$ $(6, 0)$ $(-3, -4)$ and (x, y)

How would you describe the transformation?

(ii) For the transformation defined by matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, work out the image of each of the following points.

$(2, 1)$ $(-4, 3)$ $(0, 4)$ $(-5, -1)$ and (x, y)

How would you describe the transformation?

3 The identity matrix

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the identity matrix **I**.

When multiplied by another matrix **A**: **AI** = **IA** = **A**.

When **I** is used as a transformation matrix, no movement occurs.

Exercise 9B

① Work out the image of point (4, 2) for the transformation defined by matrix $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

② Work out the image of point (1, −3) for the transformation represented by matrix $\begin{bmatrix} 0 & -3 \\ -1 & 5 \end{bmatrix}$.

③ Work out the image of point (−2, −3) for the transformation defined by matrix $\begin{bmatrix} -2 & -3 \\ 2 & -1 \end{bmatrix}$.

④ The image of point (4, 3) under the transformation matrix $\begin{bmatrix} 2 & 1 \\ c & 3 \end{bmatrix}$ is (11, 1). Work out the value of c.

⑤ The image of point (a, 1) under the transformation matrix $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ is (7, 17). Work out the value of a.

⑥ The image of point (3, −2) under the transformation matrix $\begin{bmatrix} a & 2a \\ b & 3 \end{bmatrix}$ is (b, b). Work out the values of a and b.

⑦ The transformation matrix $\begin{bmatrix} 2c & d \\ c & -d \end{bmatrix}$ maps the point (2, 5) to the point (−6, 12). Work out the values of c and d.

⑧ Given that $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, show that $\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

> $\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$, i.e. **AA**

⑨ Given that $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 7 & p \end{bmatrix} = \mathbf{I}$, work out the value of p.

PS ⑩ Under the transformation matrix $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, point D is mapped to (11, 10). Work out the coordinates of D.

PS ⑪ The matrices $\mathbf{M} = \begin{bmatrix} a & 3 \\ 2 & b \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} a+1 & b+2 \\ -a & 2 \end{bmatrix}$ satisfy the equation $\mathbf{MN} = c\mathbf{I}$. Work out the values of a, b and c.

PS ⑫ (i) Under the transformation matrix $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, which of the following points is **not** invariant?

(1, 3) (2, 4) (3, 9) (5, 15)

(ii) If the point (x, y) is invariant under $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, identify a condition for all of the invariant points of this transformation.

Note

Question 12 is for interest only as this specification does not assess a candidate's knowledge of invariant points, i.e. a point which does not move (see GCSE Maths).

FUTURE USES

At A-Level, invariant points (and invariant lines) are investigated to a much greater depth.

4 Transformations of the unit square

The unit square has vertices O(0, 0), A(1, 0), B(1, 1) and C(0, 1).

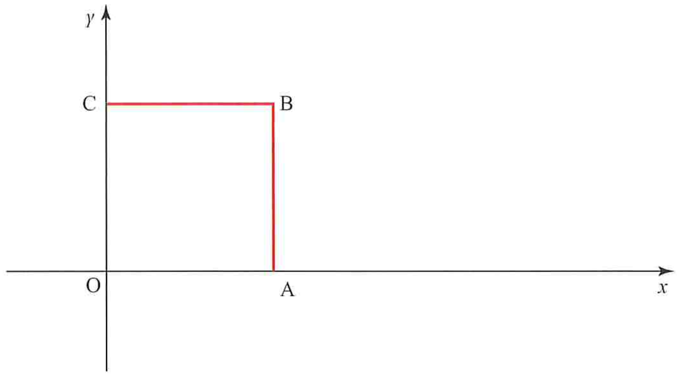


Figure 9.1

The following transformations of the unit square are the only ones that you will be expected to carry out.

Reflections

in the x-axis in the y-axis in the line $y = x$ in the line $y = -x$

Rotations about the origin

through 90° through 180° through 270°

A rotation of 270° is the same as a rotation of 90° clockwise.

Note

Rotations are anticlockwise unless the question states otherwise.

Enlargements, centre the origin

with positive scale factors with negative scale factors

You will need to be able to work out the matrix that represents any of the above transformations.

Example 9.6

Work out the 2 × 2 matrix that represents each of the following transformations.

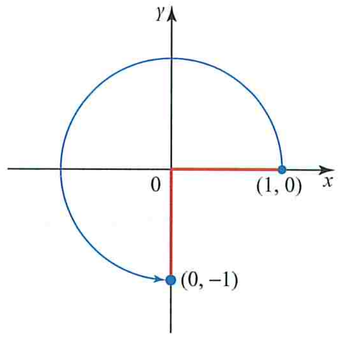
(i) Rotation through 270° about the origin.

(ii) Enlargement of scale factor 3, centre the origin.

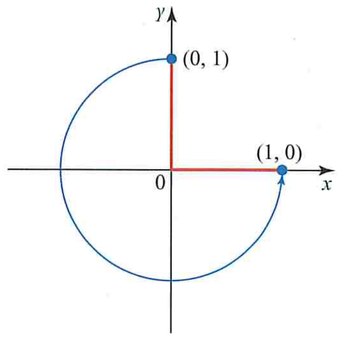
Solution

In both parts, consider the images of the position vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(i)



A rotation of 270° maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$



A rotation of 270° maps $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Figure 9.2

Write these two position vectors side by side: $\begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

to give the transformation matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(ii)

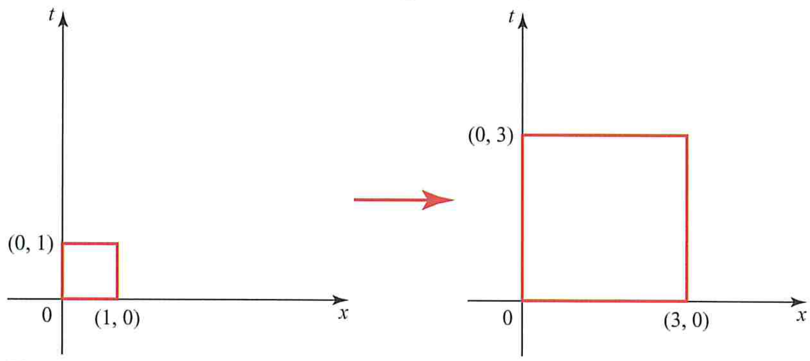


Figure 9.3

Point (1, 0) maps to (3, 0), which has position vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Point (0, 1) maps to (0, 3), which has position vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

Write these two position vectors side by side: $\begin{bmatrix} 3 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

to give the transformation matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Example 9.7

The unit square OABC is transformed by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ to OA′B′C′.

Show the image on a diagram, labelling each vertex.

Solution

Method 1

Use the matrix to work out the image of each vertex.

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Image of O(0, 0) is O(0, 0).

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ Image of A(1, 0) is A′(−1, 0).

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ Image of B(1, 1) is B′(−1, −1).

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ Image of C(0, 1) is C′(0, −1).

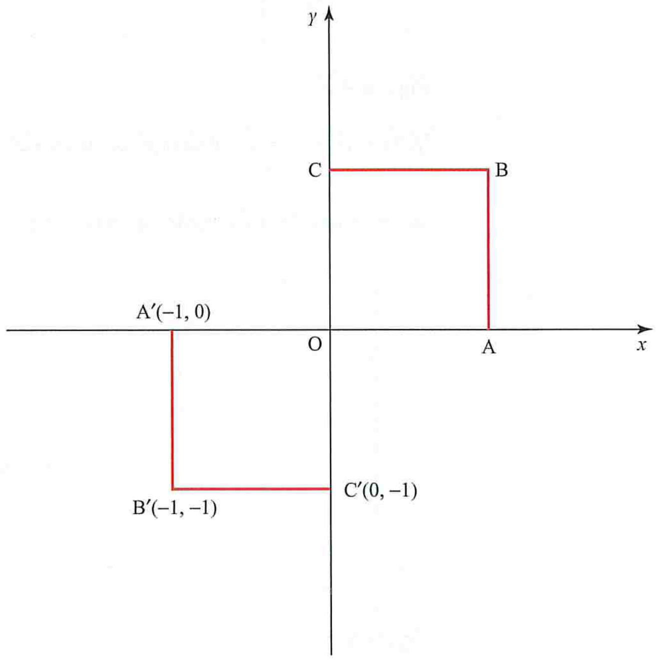


Figure 9.4

Method 2 (alternative method)

If the matrix is recognised as representing a rotation through 180°, the diagram can be drawn without first calculating the points.

Exercise 9C

① Work out the 2×2 matrix that represents each of the following transformations.

(i) Reflection in the x-axis.

(ii) Rotation of 90° about O.

(iii) Enlargement, scale factor 2, centre the origin.

(iv) Reflection in the y-axis.

(v) Reflection in the line $y = x$.

(vi) Rotation by 180°, centre the origin.

(vii) Reflection in the line $y = -x$.

(viii) Enlargement, scale factor −3, centre O.

(ix) Enlargement, centre O, scale factor $\frac{1}{2}$.

② The unit square OABC is transformed by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to OA′B′C′. Show the image on a diagram, labelling each vertex.

③ The unit square OABC is transformed by the matrix $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ to OA′B′C′.

Show the image on a diagram, labelling each vertex.

④ Describe fully the transformations given by the following matrices.

(i) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

⑤ The unit square OABC is transformed to OA′B′C′.

OA′B′C′ is shown on the diagram.

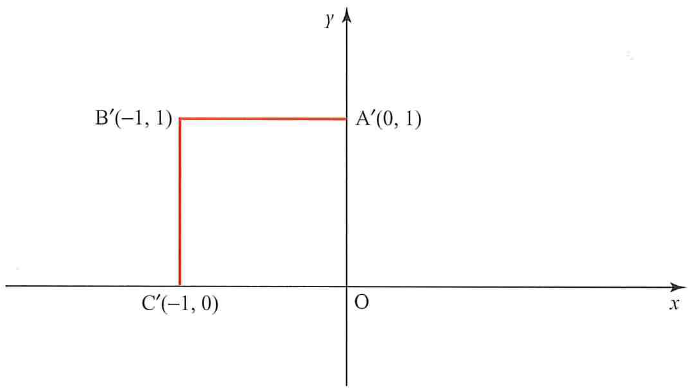


Figure 9.5

Work out the matrix for the transformation.

⑥ The unit square OABC is transformed by the matrix $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ to OA′B′C′.

Work out the area of OA′B′C′.

⑦ The unit square OABC is transformed by the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ to OA′B′C′.

The area of OA′B′C′ is 64 square units.

Work out the two possible values of k.

Note

Question 8 is for interest only as this specification only includes rotations of 90°, 180° and 270°.

⑧ (i) Draw a diagram to show the unit square OABC rotated 45° about the origin.

(ii) Work out the coordinates of A′ and C′ (the images of A and C).

(Hint: $\sin 45° = \frac{\sqrt{2}}{2}$ and $\cos 45° = \frac{\sqrt{2}}{2}$.)

(iii) Hence write down the transformation matrix for a rotation of 45° about the origin.

5 Combining transformations

Two transformations may be applied successively.

The two transformations can be combined into a single transformation that maps point A to point A″.

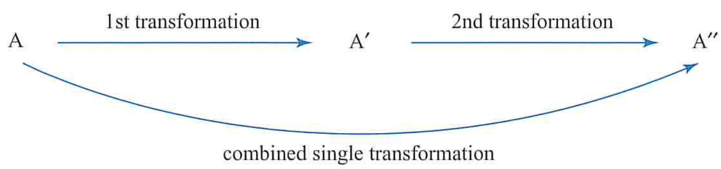


Figure 9.6

When a transformation represented by matrix **P** is followed by a transformation represented by matrix **Q**, the matrix for the combined transformation is **QP**.

Example 9.8

Point L is transformed by the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ to the point M.

Point M is then transformed by the matrix $\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$ to the point N.

Work out the matrix that transforms point L to point N.

Solution

Note that the matrix for the second transformation is written first when working out the combined transformation matrix.

Multiply the matrices in the correct order.

$$\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 1 \end{bmatrix}$$

Example 9.9

The unit square is transformed by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ followed by a further transformation by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Work out the matrix for the combined transformation, and describe it geometrically.

Solution

Multiply the matrices in the correct order.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

This is the transformation matrix for which (1, 0) maps to (0, −1), and (0, 1) maps to (−1, 0).

Discussion point

Can you identify a pair of transformations (described geometrically) for which the result is the same regardless of the order in which they are applied?

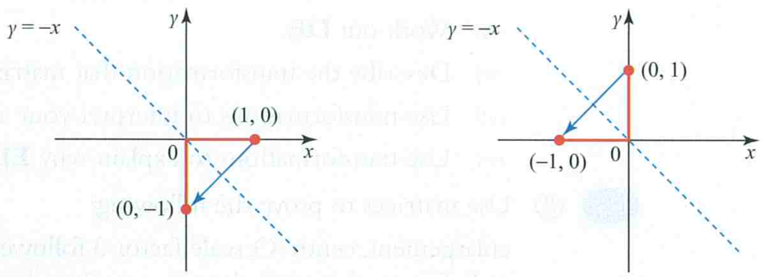


Figure 9.7

The combined matrix represents a reflection in the line $y = -x$.

Exercise 9D

① Point P(3, −2) is transformed by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, followed by a further transformation $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the x-coordinate of the image point of P.

② Point W(−1, 4) is transformed by $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$, followed by a further transformation $\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the coordinates of the image point of W.

③ The unit square is reflected in the x-axis followed by a rotation through 180°, centre the origin.

Work out the matrix for the combined transformation.

④ The unit square is enlarged, centre the origin, scale factor 2, followed by a reflection in the line $y = x$.

Work out the matrix for the combined transformation.

⑤ The unit square is rotated by 90°, centre the origin, followed by a reflection in the y-axis.

Work out the matrix for the combined transformation.

⑥ Matrix $\mathbf{P} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(i) Describe the transformation that this matrix represents.

(ii) Work out $\mathbf{P}^2$.

(iii) Use transformations to interpret your answer to part (ii).

⑦ Matrix $\mathbf{D} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Matrix $\mathbf{E} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(i) Describe the transformation that matrix **D** represents.

(ii) Describe the transformation that matrix **E** represents.

(iii) Work out **DE**.

(iv) Describe the transformation that matrix **DE** represents.

(v) Use transformations to interpret your answer to part (iv).

(vi) Use transformations to explain why **ED** = **DE**.

PS ⑧ Use matrices to prove the following:

enlargement, centre O, scale factor 3 followed by enlargement, centre O, scale factor −2 is equivalent to a single enlargement, centre O, scale factor k, where k is an integer to be found.

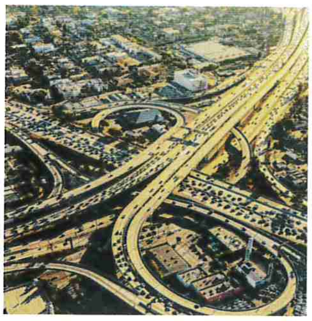
PS ⑨ (i) Calculate the combined transformation matrix of $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, followed by $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, and finally $\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$.

(ii) Under this combined transformation, point A is mapped to the point $(-14, 7)$. Work out the coordinates of point A.

PS ⑩ Identify three matrices **A**, **B** and **C** such that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

FUTURE USES

At A-Level (Further Maths only) 3 × 3 matrices are used to describe 3-dimensional transformations. They can also be used to solve simultaneous equations in three unknowns.



REAL-WORLD CONTEXT

Matrices are used in discrete mathematics to represent networks for a variety of situations such as water or traffic flow, or a rail network, e.g. the London Underground. They can also be used to solve problems which require costs to be minimised, or profits to be maximised. You will meet these applications if you study mathematics at A-Level.

Matrices have applications in most scientific fields, including quantum mechanics and electromagnetism. Computer programmers also make use of matrices, e.g. when coding graphics.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- multiply a 2×2 matrix by a scalar
- multiply a 2×2 matrix by a 2×1 matrix
- multiply a 2×2 matrix by a 2×2 matrix
- recognise the identity matrix
- recognise and use the following transformations when written as matrices:
  - reflection in the x-axis
  - reflection in the y-axis
  - reflection in the line $y = x$
  - reflection in the line $y = -x$
  - rotation of $90°$ about the origin
  - rotation of $180°$ about the origin
  - rotation of $270°$ about the origin
  - enlargement of scale factor k, centred on the origin
- apply a combination of transformations
- work out the matrix which represents a combination of transformations.

KEY POINTS

1 To multiply a 2×2 matrix by a 2×1 matrix, each row of the 2×2 matrix is multiplied by the column of the 2×1 matrix.

To multiply a 2×2 matrix by a 2×2 matrix, each row of the first 2×2 matrix is multiplied by each column of the second matrix.

2 If two matrices are equal, the corresponding elements can be equated.

3 A point P(x, y) can be transformed to an image point P′(x', y').

If the transformation matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

4 $\mathbf{I}$ is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5 A transformation of matrix **A** followed by a transformation of **B** is a combined transformation. The matrix for the combined transformation is **BA**.

Key words

Calculate	Work out the numerical value (often used after making a substitution)
Draw (a graph)	Draw axes on graph paper, plot points accurately and join with a straight line or smooth curve
Evaluate	Give a numerical value for your answer
Expand	Remove brackets
Expand and simplify	Remove brackets and collect like terms
Explain	Give reasons, either in words, or using mathematical symbols, or both
Expression	One or more terms, for example one side of a formula
Factorise	Write as a product
Give your answer in its simplest form	Cancel answers given as ratios or fractions or collect like terms
Hence	Use earlier work to deduce the result
Hence or otherwise	Using previous work to deduce the result is an option
Plot	Mark points (usually on graph paper) and join with a straight line or curve
Prove	Show all relevant steps (include explanations of facts used in geometrical proofs)
Show that	Show all relevant steps to reach a given result
Sketch (a graph)	Do not use graph paper. Draw axes and show the correct shape in each quadrant. Label appropriate points (e.g. intersection with axes, stationary points)
Verify	'Check out' a statement or result that you have been given
Work out the exact value of	Give the answer as an integer, fraction, recurring decimal, in terms of π, etc. or a surd
Write down	The answer should be obvious (no working is necessary)

Practice questions 1

Calculator NOT allowed $1\frac{3}{4}$ **hours**

① $\begin{bmatrix} 4 & p \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

Work out the value of p. [3 marks]

② Simplify $\dfrac{2x - 6}{x^2 + 5x - 24}$. [3 marks]

③ A triangle has sides 3 cm, 4 cm and 5 cm. The vertices of the triangle lie on the circle. Work out the exact area inside the circle not covered by the triangle. [4 marks]

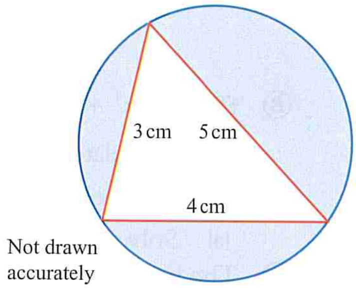


Figure 1

④ A palindromic number reads the same backwards as forwards. For example, 20 502 and 351 153 are palindromic. How many seven-digit palindromic numbers are odd? [2 marks]

⑤ The graph of $y = ab^x$ passes through the points (1, 6) and (2, 12).

(a) Work out the values of a and b. [2 marks]

(b) Sketch the graph of $y = ab^x$. [2 marks]

⑥ A, B and C lie on a circle. BD is a tangent. AC is parallel to BD.

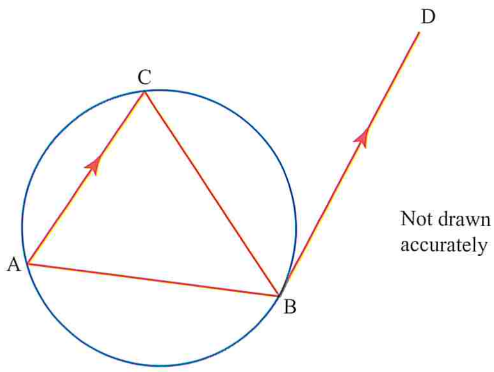


Figure 2

Prove that triangle ABC is isosceles. [3 marks]

⑦ A circle has centre C and passes through A(−7, 3) and B(5, 3), as shown in the diagram.

AC = BC = 10

Work out the equation of the circle. [4 marks]

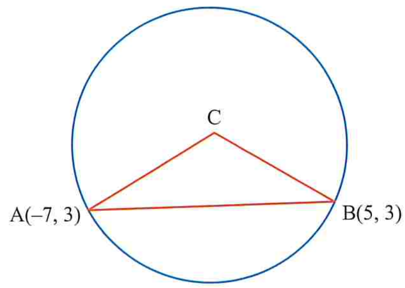


Figure 3

⑧ $f(x) = 2x^4 + x^3 - 7x^2 + 5x - 1$

(a) Calculate f(1). [1 mark]

(b) Use the factor theorem to show that $(2x - 1)$ is a factor of $f(x)$. [2 marks]

(c) Solve $f(x) = 0$. [4 marks]

⑨ The line $3y + x = 10$ and the circle $x^2 + y^2 = 20$ intersect at P and Q.

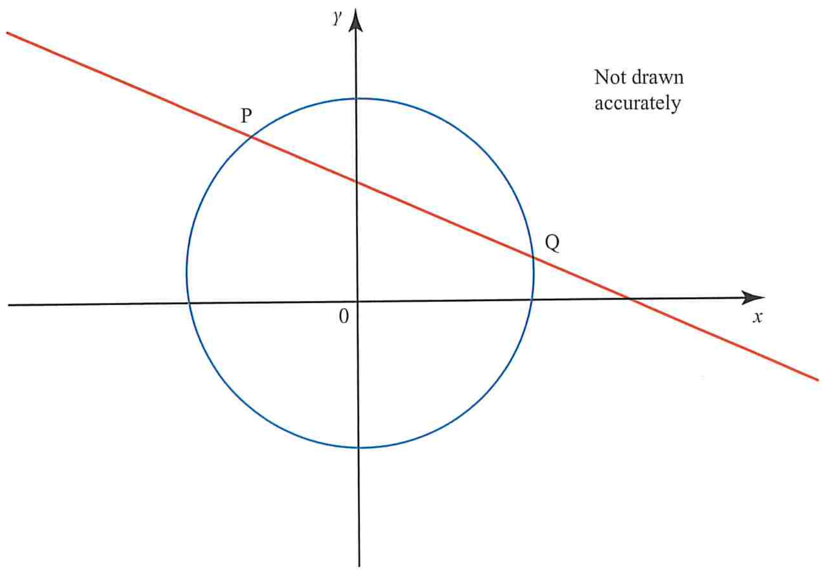


Figure 4

Work out the coordinates of P and Q. [6 marks]

⑩ Rationalise the denominator of $\dfrac{4 - \sqrt{3}}{\sqrt{3} + 1}$. [3 marks]

⑪ (a) Write $\sqrt{\dfrac{x^{\frac{4}{3}} \times x^{\frac{8}{3}}}{x}}$ as a single power of x. [2 marks]

(b) Solve $x^{-\frac{1}{3}} = \dfrac{2}{5}$. [2 marks]

⑫ $(x+4)^2 + p \equiv x^2 + qx + 28$

Work out the values of p and q. [3 marks]

⑬ (a) Sketch the graph of $y = \cos x$ for $-180° \leqslant x \leqslant 540°$. [2 marks]

(b) Solve $\cos x$ $x = \frac{\sqrt{3}}{2}$ for $-180° \leqslant x \leqslant 540°$. [3 marks]

⑭ A sequence has nth term $= \frac{3n+2}{1-6n}$.

Prove that the limiting value of the nth term as $n \to \infty$ is $-\frac{1}{2}$. [3 marks]

⑮ Work out the equation of the tangent to the curve $y = (x+5)(x-3)$ at the point where $x = -3$. Give your answer in the form $y = mx + c$. [5 marks]

⑯ The nth term of a sequence is $an^2 + bn + c$. The 1st, 5th and 7th terms are −2, 2 and 16 respectively.

(a) Substituting $n = 1, 5$ and 7, write three equations in a, b and c. [2 marks]

(b) Solve your equations to work out the nth term of the sequence. [5 marks]

⑰ (a) Factorise $y^2 - 3y + 2$ [2 marks]

(b) Hence solve $y^2 - 3y + 2 = 0$ [1 mark]

(c) Hence solve $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2 = 0$ [2 marks]

⑱ $f(x) = x^2 + 6, \ x \geqslant 0$

$g(x) = 2x + 1$, for all x

(a) Work out the inverse function f^{-1} and its domain. [2 marks]

(b) Solve $fg(x) = gf(x)$. [4 marks]

⑲ $14xy^6$ is one of the terms in the expansion of $(ax + y)^n$.

(a) Write down the value of n. [1 mark]

(b) Calculate the value of a. [2 marks]

Practice questions 2

Calculator allowed $1\frac{3}{4}$ **hours**

① Which of the following equations matches the graph shown in Figure 1? [1 mark]

$y = x^2 - 3$ $y = x^2 + 3$ $y = (x - 3)^2$ $y = (x + 3)^2$

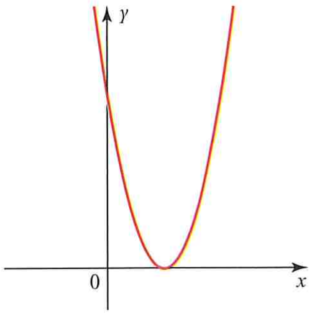


Figure 1

② Simplify $\frac{6xy^2}{8x^2} \div \frac{10x^3y}{12y^4}$. [2 marks]

③ Expand and simplify $(x^2 - 3x + 4)(2x^2 + 5x - 7)$. [2 marks]

④ Work out the equation of the line that is perpendicular to $2x + 3y = 5$ and crosses the x-axis at $(-5, 0)$. Write your answer in the form $ax + by = c$. [3 marks]

⑤ $y = 4x^3 + x^2 - 7x$

(a) Work out $\frac{dy}{dx}$. [2 marks]

(b) Work out the rate of change of y with respect to x when $x = 1.5$ [2 marks]

⑥ C is the centre of the circle, shown in Figure 2. P is a point on the circumference.

Work out the value of x. [3 marks]

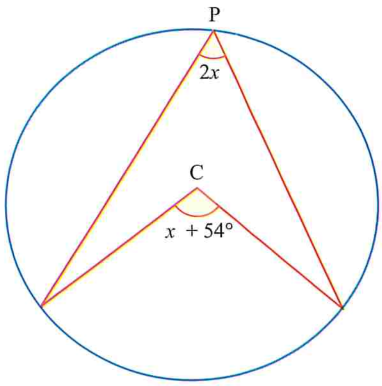


Figure 2

⑦ h increased by 50% is equal to 60% of m.

Express the ratio $h : m$ in its simplest terms. [2 marks]

⑧ A ship travels for 25 km on a bearing of 065° from A to B.

It then travels for 18 km from B to C on a bearing of 135°.

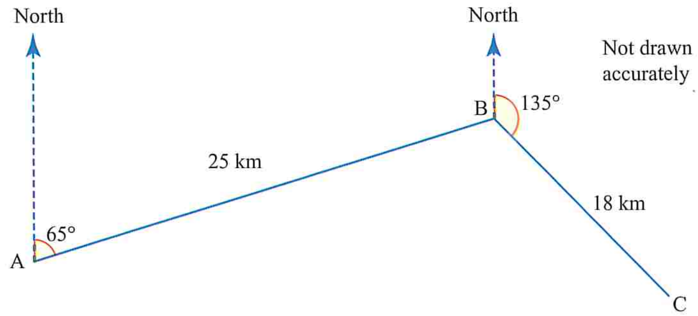


Figure 3

(a) Work out the distance AC. [3 marks]

(b) Work out the bearing of C from A. [4 marks]

⑨ Figure 4 shows the straight line PQR.

PR = 5PQ

Work out the coordinates of Q. [4 marks]

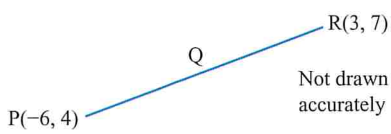


Figure 4

⑩ Draw the graph of $y = g(x)$ where

$$g(x) = \begin{cases} 3x & 0 \leqslant x \leqslant 2 \\ 6 & 2 < x \leqslant 5 \\ 16 - 2x & 5 < x \leqslant 8 \end{cases}$$

[3 marks]

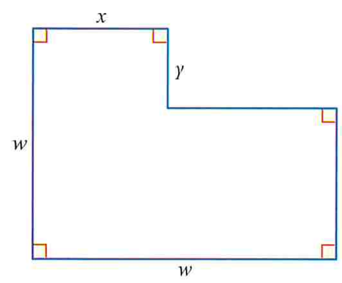


Figure 6

⑪ Figure 6 shows an L-shape. (All lengths are in centimetres.)

The shape has area A cm².

(a) Show that $A = w^2 + xy - wy$. [2 marks]

(b) Rearrange the formula to make y the subject. [3 marks]

⑫ Figure 7 shows a cyclic quadrilateral. Each angle of the quadrilateral is given in terms of x and y.

Work out x and y. [5 marks]

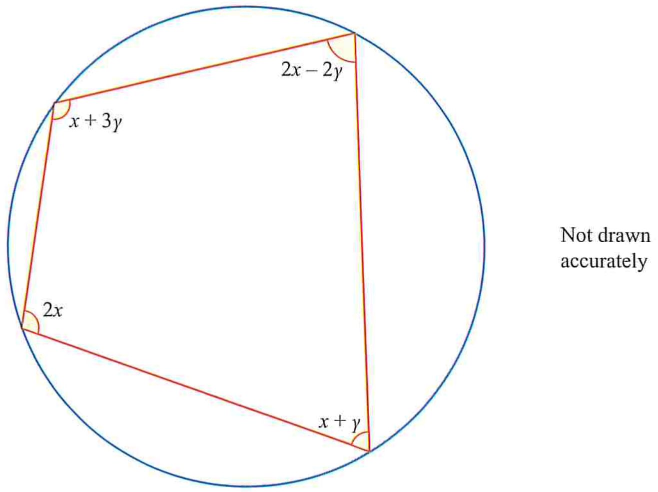


Figure 7

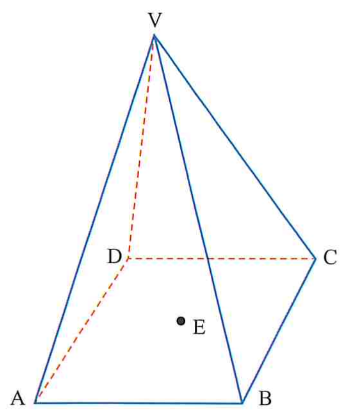


Figure 8

⑬ A paperweight is in the shape of a square-based pyramid (see Figure 8). The base ABCD has edge 5 cm. The vertex V is directly above the centre of the base, E.

VE = 7 cm.

(a) Work out VA. [3 marks]

(b) Work out the angle between VA and ABCD. [3 marks]

(c) Work out the angle between VAB and ABCD. [3 marks]

⑭ A triangle and a rectangle are shown in Figure 9. All dimensions are in centimetres.

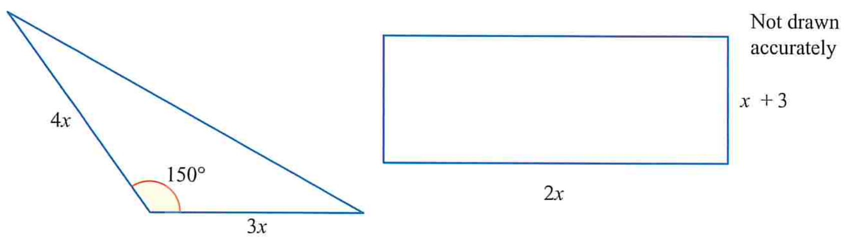


Figure 9

(a) Show that the area of the triangle is $3x^2$ cm². [2 marks]

(b) Work out the range of values of x for which area of triangle < area of rectangle. [5 marks]

⑮ Point P is transformed by $\begin{bmatrix} 3 & 1 \\ 6 & 3 \end{bmatrix}$ followed by a further transformation $\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$.

(a) Work out the matrix for the combined transformation. [2 marks]

(b) The image of point P after the combined transformation is (2, 3). Work out the coordinates of P. [3 marks]

⑯ Prove that $\sin x \tan x \equiv \frac{1}{\cos x} - \cos x$. [3 marks]

⑰ The square-based open container in Figure 10 is a cuboid with a volume of 1000 cm^3.

The open end is opposite the square base as shown.

Each side of the square base is x cm, and the height of the container is h cm.

(a) Write h in terms of x. [1 mark]

(b) Hence show that the container's external surface area, A, is given by

$A = x^2 + \frac{4000}{x}$. [2 marks]

(c) Use calculus to work out the container's minimum surface area. [3 marks]

(d) Use the second derivative to justify that your answer to part (c) is a minimum. [3 marks]

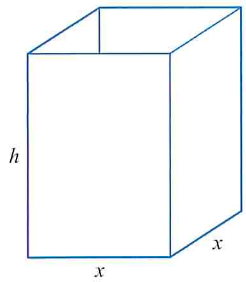


Figure 10

⑱ A kite and a sector of a circle are shown in Figure 11.

The two shapes share the sides OA and OC, where O is the centre of the circle, and A and C are points on the circle.

The radius of the circle is 7 cm and the angle AOC is 150° as shown.

The other two sides, AB and CB, of the kite are tangents to the circle.

Calculate the total area of the shape. [7 marks]

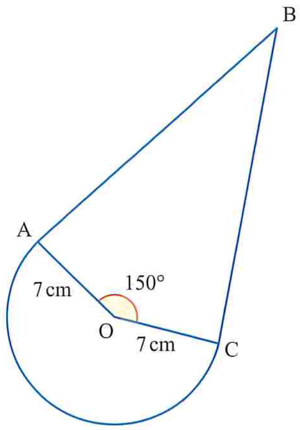


Figure 11

Answers

Chapter 1

Exercise 1A (page 3)

1 (i) 5:2
(ii) 2:3
(iii) 11:2
(iv) 1:2
(v) 3:1

2 (i) £69
(ii) 260
(iii) 11.2 cm

3 (i) 7163
(ii) 6.72
(iii) £6.30

4 (i) 84
(ii) £420
(iii) £12
(iv) 12

5 (i) 741
(ii) 3136.25
(iii) £1314
(iv) £48.85

6 (i) $\frac{52}{45}$ or $1\frac{7}{45}$
(ii) $\frac{1}{32}$
(iii) $\frac{53}{20}$ or $2\frac{13}{20}$

7 (i) 10.7
(ii) 4.9
(iii) 0.04
(iv) 0.60

8 35

9 80

Discussion point (page 5)

Factorise means the expression must be written as a product of factors.

Factorise *fully* means that each factor cannot be factorised any further.

Exercise 1B (page 5)

1 (i) $10a - b - 2c$
(ii) $6x - 3y - 4z$
(iii) $19x + 5y$
(iv) $p + 14q$
(v) $5x$
(vi) $2a^2 + 12a - 12$
(vii) $3q^2 - 3p^2$
(viii) $10fg + 10fh - 5gh$

2 (i) $2(4 + 5x^2)$
(ii) $2b(3a + 4c)$
(iii) $2a(a + 2b)$
(iv) $pq(q^2 + p^2)$
(v) $3xy(x + 2y^3)$
(vi) $2pq(3p^2 - 2pq + q^2)$
(vii) $3lm^2(5 - 3l^2m + 4lm^2)$
(viii) $12a^4b^4(7a - 8b)$

3 (i) $4(5x - 4y)$
(ii) $6(x + 1)$
(iii) $z(x - y)$
(iv) $2q(p - r)$
(v) $k(l + n)$
(vi) $-4(a + 2)$
(vii) $3(x^2 + 2y^2)$
(viii) $2(a + 4)$

4 (i) $10a^3b^4$
(ii) $12p^3q^4r$
(iii) lm^2n^2p
(iv) $36r^4s^3$
(v) $64ab^2c^2d^2e$
(vi) $60x^3y^3z^3$
(vii) $84a^5b^9$
(viii) $42p^3q^8r^5$

5 (i) $2a$
(ii) pq
(iii) $\frac{4b}{a}$
(iv) $\frac{bd}{ac}$
(v) $\frac{2xy^2z}{3}$
(vi) $\frac{5}{2a^2b^2}$
(vii) $\frac{7p^2r^3}{6q^4s^2}$

6 (i) $\frac{11a}{12}$
(ii) $\frac{13x}{20}$
(iii) $\frac{7p}{12}$
(iv) $\frac{s}{3}$
(v) $\frac{5b}{12}$
(vi) $\frac{7a}{3b}$
(vii) $\frac{(5q - 3p)}{2pq}$
(viii) $-\frac{5x}{6y}$

7 (i) $16x - 10$
(ii) $7x^2 + x - 3$

8 (i) $16x + 2$
(ii) $11x^2 + x - 2$
(iii) $x(3x + 2)(x - 1)$ or $3x^3 - x^2 - 2x$

Discussion points (page 7)

An equation is used to show that two expressions or numbers are equal, e.g. $5x + 2 = 3x + 8$. An equation must contain an equals sign.

Solving an equation finds the values of any unknown(s) which satisfy the equation.

Discussion point (page 7)

The letter and number do not have to change sides, but convention puts the letter on the left. This also makes it easier to read, as we read from left to right.

Exercise 1C (page 8)

1 (i) $x = 7$
(ii) $a = -2$
(iii) $x = 2$
(iv) $y = 2$
(v) $c = 5$
(vi) $p = 10$
(vii) $x = -5$
(viii) $x = -6$
(ix) $y = 7$
(x) $k = 42$
(xi) $t = 60$
(xii) $p = -55$
(xiii) $p = 0$

2 (i) $2l + 2(l - 80) = 600$
(ii) $l = 190$; Area $= 20\,900\,\text{m}^2$

3 (i) $2(s + 4) + s = 17$
(ii) Ben = 7, Chris = 7, Stephen = 3

4 (i) $5c - a$
(ii) $5c - 15 = 40$; $c = 11$

5 (i) $3j + 12$ or $2(j + 12)$
(ii) $3j + 12 = 2(j + 12)$; $j = 12$

6 (i) $8a$
(ii) $6a + 6$
(iii) $a = 3$

7 (i) $m - 2, m - 1, m, m + 1, m + 2$
(ii) $m - 2 + m - 1 + m + m + 1 + m + 2 = 105$; $m = 21$
(iii) 19, 20, 21, 22, 23

8 (i) $2(x + 2) = 5(x - 3)$; $x = 6\frac{1}{3}$
(ii) $16\frac{2}{3}\,\text{cm}^2$

Discussion point (page 9)

Let the price of a large ice cream be l and the price of a small be s. Form two simultaneous equations in l and s using the two pieces of information given, and solve.

A small ice cream costs 80p.
A large ice cream costs £1.20.

Discussion point (page 9)

The second sentence duplicates the information given in the first sentence.

So there is an infinite number of solutions.

Exercise 1D (page 11)

1 (i) $0.3b$ or $\frac{3b}{10}$
(ii) $\frac{9y}{2}$
(iii) $\frac{cd}{100}$

2 $66\frac{2}{3}\%$

3 (i) $1.2a$
(ii) $1.05b$
(iii) $0.65k$
(iv) $0.98m$

4 $1.8a = 1.5b$
$\frac{1.8}{1.5} = \frac{b}{a}$
$1.2 = \frac{b}{a}$

5 60%

6 8 : 12 : 27

7 (i) $a = \frac{5b}{2}$
(ii) 6 : 1
(iii) 5 : 4

8 15 : 8

9 24 : 25

10 $100 + m : 100 - m$

11 40 boys and 50 girls

Discussion point (page 12)

When you multiply out the double bracket, there are terms in x^2 and x and a number, but no other terms (e.g. no x^3, no $\sqrt{x}$, no $\frac{1}{x}$).

Exercise 1E (page 13)

1 (i) $x^2 + 9x + 20$
(ii) $x^2 + 4x + 3$
(iii) $2a^2 + 9a - 5$
(iv) $6p^2 + 5p - 6$
(v) $x^2 + 6x + 9$
(vi) $4x^2 - 9$
(vii) $14m - 3m^2 - 8$
(viii) $12 + 4t - 5t^2$
(ix) $16 - 24x + 9x^2$
(x) $m^2 - 6mn + 9n^2$

2 (i) $x^5 - x^4 + 2x^3 - 3x^2 + x - 2$
(ii) $x^6 + 2x^5 - 3x^4 - 4x^3 + 5x^2 + 6x - 3$
(iii) $2x^5 - 4x^4 - x^3 + 11x^2 - 13x + 5$
(iv) $x^6 - 1$
(v) $x^3 + 4x^2 + x - 6$
(vi) $2x^3 + 5x^2 - 14x - 8$
(vii) $x^3 + 3x^2 + 3x + 1$
(viii) $p^3 - 15p^2 + 75p - 125$
(ix) $8a^3 + 36a^2 + 54a + 27$
(x) $2x^3 - 17x - 18$
(xi) $2x^2 - 2x$

3 (i) $2x^3 - 5x^2 - 3x$
(ii) $10x^2 - 14x - 6$

4 (i) $3x^3 - \frac{1}{2}x^2 - \frac{21}{2}x + 5$
(ii) $11x^2 - 15x - 4 + 4xy - 2y$

5 (i) $a^2 + 2ab + b^2$
(ii) $a^3 + 3a^2b + 3ab^2 + b^3$
(iii) $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
(iv) $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

6 (i) $x^3 + 18x^2 + 108x + 216$
(ii) $p^3 - 6p^2 + 12p - 8$
(iii) $16y^4 + 32y^3 + 24y^2 + 8y + 1$
(iv) $x^4 - 12x^3 + 54x^2 - 108x + 81$
(v) $243w^5 - 1620w^4 + 4320w^3 - 5760w^2 + 3840w - 1024$

Activity 1.1 (page 14)

(i) 7
(ii) 1
(iii) 6

(iv) $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Exercise 1F (page 17)

1 (i) $1 + 3x + 3x^2 + x^3$

(ii) $y^4 + 4y^3 + 6y^2 + 4y + 1$

(iii) $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

(iv) $125 + 75w + 15w^2 + w^3$

(v) $p^4 + 16p^3 + 96p^2 + 256p + 256$

(vi) $32 + 80m + 80m^2 + 40m^3 + 10m^4 + m^5$

2 (i) $x^3 - 3x^2y + 3xy^2 - y^3$

(ii) $1 - 8x + 24x^2 - 32x^3 + 16x^4$

(iii) $32 - 80y + 80y^2 - 40y^3 + 10y^4 - y^5$

(iv) $125 - 150p + 60p^2 - 8p^3$

(v) $81x^4 - 432x^3 + 864x^2 - 768x + 256$

(vi) $1024x^5 - 1280x^4 + 640x^3 - 160x^2 + 20x - 1$

3 $1 + 6x + 15x^2$

4 $x^7 + 14x^6 + 84x^5$

5 -4320

6 10

7 1

8 36

9 $81x^8 + 108x^5 + 54x^2 + \frac{12}{x} + \frac{1}{x^4}$

10 (i) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

(ii) $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

(iii) $20x + 160x^3 + 64x^5$

11 (i) $27 + 27w + 9w^2 + w^3$

(ii) $27 + 27x + 54y + 9x^2 + 36xy + 36y^2 + x^3 + 6x^2y + 12xy^2 + 8y^3$

12 (i) 6

(ii) ± 4

(iii) 3840

13 160

14 3360

Activity 1.2 (page 18)

(i) They are all powers of 11

(ii) 2^n

(iii) n

(iv) $\frac{1}{2}n(n-1)$

(v) Palindromic

Future uses (page 18)

${}^nC_r$ is the $(r + 1)$th number in the nth row of Pascal's triangle

Note: Candidates will not be expected to have knowledge of the ${}^nC_r$ function.

Discussion point (page 19)

A rational number is a number which can be expressed in the form $\frac{a}{b}$ where a and b are integers. So a rational number can be a fraction or an integer (when $b = 1$); and can be positive or negative.

Exercise 1G (page 20)

1 (i) $4\sqrt{2}$

(ii) $5\sqrt{5}$

(iii) $5\sqrt{3}$

(iv) $\sqrt{2}$

(v) $3\sqrt{3}$

(vi) $7\sqrt{2} - 3$

(vii) $10\sqrt{2}$

(viii) $36 + 3\sqrt{3}$

(ix) $16\sqrt{5}$

(x) $\sqrt{3}$

2 (i) $3 - 2\sqrt{2}$

(ii) $3 + 2\sqrt{5}$

(iii) $3\sqrt{7} - 9$

(iv) 2

(v) $11 - \sqrt{2}$

(vi) $5 - 3\sqrt{7}$

(vii) $24 - 13\sqrt{3}$

(viii) $8 - 2\sqrt{15}$

(ix) $13\sqrt{2} - 17$

(x) $17 + 12\sqrt{2}$

3 (i) $\frac{\sqrt{3}}{3}$

(ii) $\sqrt{5}$

(iii) $\frac{4\sqrt{6}}{3}$

(iv) $\frac{\sqrt{6}}{3}$

(v) 1

(vi) $\frac{\sqrt{21}}{7}$

(vii) $3\sqrt{7}$

(viii) $\frac{\sqrt{5}}{3}$

(ix) $\frac{\sqrt{15}}{5}$

(x) $\frac{\sqrt{2}}{2}$

4 (i) $\frac{12}{7}$

(ii) $4 + 8\sqrt{3}$

(iii) $20 - 4\sqrt{6}$

(iv) $\frac{12 - 5\sqrt{3}}{13}$ or $\frac{12}{13} - \frac{5}{13}\sqrt{3}$

5 (i) $45 + 29\sqrt{2}$
(ii) $38 + 17\sqrt{5}$
(iii) $97 - 56\sqrt{3}$
(iv) $73 + 28\sqrt{6}$
(v) $176 + 80\sqrt{5}$
(vi) $682 - 305\sqrt{5}$

6 (i) $v = 2$
(ii) $w = \frac{5}{3}$
(iii) $y = \frac{3}{2}$
(iv) $x = \frac{3}{5}$

7 $1351 + 780\sqrt{3}$

8 (i) $m = \frac{5}{2}$
(ii) $n = \frac{16}{5}$
(iii) $x = 6$
(iv) $x = -\frac{144}{7}$

9 $x = \frac{4\sqrt{2}}{3}$

10 $8 + \sqrt{48}$

Exercise 1H (page 22)

1 (i) $\frac{10\sqrt{3} - 2\sqrt{6}}{23}$
(ii) $\frac{4\sqrt{7} + \sqrt{14}}{14}$
(iii) $\frac{9 - 3\sqrt{3}}{2}$
(iv) $\frac{8 + 5\sqrt{2}}{7}$
(v) $\frac{\sqrt{7} - 2}{3}$
(vi) $10\sqrt{3} + 3 - 10\sqrt{2} - \sqrt{6}$

2 $12 + 9\sqrt{2}$

3 $18\sqrt{5} - 40$

4 $1 - \frac{1}{3}\sqrt{3}$

5 $3 - \sqrt{5}$

6 $3\sqrt{3} + 2\sqrt{2}$

7 $10 - 4\sqrt{2}$

8 $1 + \sqrt{3}$

Activity 1.3 (page 23)

The factorial function is only defined for positive integers and zero.

A calculator will produce an error message if a negative number, or a non-integer, is input.

A calculator will interpret $-5!$ as $-(5!)$, giving the answer -120.

Discussion point (page 24)

The answer to part (i) can be calculated as $\frac{12!}{61} = 665280$.

So the answer to part (ii) can be written as $\frac{12!}{6! \times 6!} = 924$.

Exercise 1I (page 25)

1 120

2 5040

3 (i) 5040
(ii) 117 649

4 12

5 180

6 3 628 800

7 (i) 360
(ii) 1296

8 (i) 720
(ii) 48

9 (i) 90
(ii) 90
(iii) 1999
(iv) 100

10 120

11 40 320

12 (i) 360
(ii) 240

13 $n(n-1)(n-2) \times \ldots \ldots \times 2 \times 1$ (which could be written more simply as $n!$ but the factorial notation is not assessed in this specification)

14 (i) 120
(ii) 1287

Chapter 2

Discussion point (page 29)

Yes, except that the rows and columns could be interchanged.

Discussion point (page 29)

Yes, but the brackets will be in the reverse order. (Work it through to check for yourself.)

Discussion point (page 31)

Factorising gives $(2x - 5)(x - 3)$. The factors are the other way around.

Exercise 2A (page 32)

1 (i) $(a + d)(b - c)$
(ii) $(2x + w)(y + 1)$
(iii) $(2p - 3r)(q - 4)$
(iv) $(5 - 2n)(1 - m)$

2 (i) $(x + 2)(x + 3)$
(ii) $(y - 1)(y - 4)$
(iii) $(m - 4)^2$
(iv) $(m - 3)(m - 5)$
(v) $(x + 5)(x - 2)$
(vi) $(a + 12)(a + 8)$
(vii) $(x - 3)(x + 2)$
(viii) $(y - 12)(y - 4)$
(ix) $(k + 6)(k + 4)$
(x) $(k - 12)(k + 2)$

3 (i) $(x + 2)(x - 2)$
(ii) $(a + 5)(a - 5)$
(iii) $(3 + p)(3 - p)$
(iv) $(x + y)(x - y)$
(v) $(t + 8)(t - 8)$
(vi) $(2x + 1)(2x - 1)$
(vii) $(2x + 3)(2x - 3)$
(viii) $(2x + y)(2x - y)$
(ix) $(4x + 5)(4x - 5)$
(x) $(3a + 2b)(3a - 2b)$

4 (i) $(2x + 1)(x + 2)$
(ii) $(2a - 3)(a + 7)$
(iii) $(5p - 1)(3p + 1)$
(iv) $(3x - 1)(x + 3)$
(v) $(5a + 1)(a - 2)$
(vi) $(2p - 1)(p + 3)$
(vii) $(4x - 1)(2x + 3)$
(viii) $(2a - 9)(a + 3)$
(ix) $(3x - 5)^2$
(x) $(2x + 5)(2x - 3)$

5 (i) $(x + y)(x + 2y)$
(ii) $(x + 5y)(x - y)$
(iii) $(a - 4b)(a + 3b)$
(iv) $(c - 3d)(c - 8d)$
(v) $(x + 4y)(x + 5y)$
(vi) $(p + 5r)(p - 3r)$
(vii) $(a + 3r)(a - 5r)$
(viii) $(s - 2t)^2$
(ix) $(m - 6n)(m + n)$
(x) $(r + 4s)(r - 2s)$

6 (i) $(3a + 1)(a + 1)$
(ii) $(4x + 5)(2x - 3)$
(iii) $(3p - 2)(p - 4)$
(iv) $(2 + 5y)(6 - 5y)$
(v) $(a + 1)(3a + 1)$
(vi) $(4x + 5)(2x - 3)$
(vii) $(3p - 2)(p - 4)$
(viii) $12y - 4y^2$

7 (i) $(2x + y)(x + 2y)$
(ii) $(3x - y)(x + 2y)$
(iii) $(5a - 3b)(a - b)$
(iv) $(3c + 4d)(2c - d)$
(v) $(6p - q)(p - 6q)$
(vi) $(7g - 2h)(g + h)$
(vii) $(3h - 4k)(2h + k)$
(viii) $(4w - x)(2w - x)$

8 (i) $x(x + 2)(x - 2)$
(ii) $a^2(a + 4)(a - 4)$
(iii) $y^3(3 + y)(3 - y)$
(iv) $2x(x + 1)(x - 1)$
(v) $p^2(2p + 3)(2p - 3)$
(vi) $x(10 + x)(10 - x)$
(vii) $2c(3c + 1)(3c - 1)$
(viii) $2x(2x + 5y)(2x - 5y)$

Activity 2.1 (page 32)

(i) 81 and a^4
(ii) $(a^2)^2 - 9^2$
(iii) $(a^2 + 9)(a^2 - 9)$
$= (a^2 + 9)(a + 3)(a - 3)$

Activity 2.2 (page 33)

(i) $(5x + 3)(2x + 1)$
(ii) $(5p + 5q + 3)(2p + 2q + 1)$

Discussion point (page 33)

Omit it since length must be positive.

Exercise 2B (page 34)

1 (i) $u = v - at$
(ii) $t = \dfrac{v - u}{a}$; an equation of motion

2 $b = \dfrac{2A}{h}$; area of a triangle

3 $l = \dfrac{P - 2b}{2}$; perimeter of a rectangle

4 $r = \sqrt{\dfrac{A}{\pi}}$; area of a circle

5 $c = \dfrac{2A - bh}{h}$; area of a trapezium

6 $h = \dfrac{A - \pi r^2}{2\pi r}$; surface area of a cylinder with a base but no top

7 $l = \dfrac{\lambda e}{T}$; tension of a spring or string

8 (i) $u = \dfrac{2s - at^2}{2t}$
(ii) $a = \dfrac{2(s - ut)}{t^2}$; an equation of motion

9 $x = \dfrac{\sqrt{\omega^2 a^2 - v^2}}{\omega}$; speed of a particle on an oscillating spring

Exercise 2C (page 35)

1 $m = \dfrac{2x}{3 - x}$

2 $y = \dfrac{2x}{5 - x}$

3 $b = -\dfrac{a}{7}$

4 $h = \dfrac{S - 2\pi r^2}{2\pi r}$

5 $x = \dfrac{1 - 2y}{y - 1}$

6 $c = \dfrac{1 - 2d}{d + 3}$

7 (i) $t = \dfrac{3x}{x - 1}$
(ii) 4.5

8 (i) $p = \dfrac{2 - 3r}{2r - 3}$
(ii) -1

Activity 2.3 (page 35)

(i) (a) $(x + 3)(x + 3)$
$= x^2 + 3x + 3x + 9$
$= x^2 + 6x + 9$
(b) $y = x^2 + 6x + 9$
$y = (x + 3)^2$
$(x + 3) = \pm\sqrt{y}$
$x = \pm\sqrt{y} - 3$

(ii) (a) $(x - 5)(x - 5) + 4$
$= x^2 - 5x - 5x + 25 + 4$
$= x^2 - 10x + 29$
(b) $p = x^2 - 10x + 29$
$p = (x - 5)^2 + 4$
$p - 4 = (x - 5)^2$
$x - 5 = \pm\sqrt{p - 4}$
$x = 5 \pm \sqrt{p - 4}$

Discussion points (page 35)

A fraction in arithmetic is one number divided by another number.

The definition of a fraction in algebra is the same but with 'number' replaced by 'expression'.

Discussion points (page 35)

You can cancel a fraction in arithmetic when the numerator and denominator have a common factor.

It is the same for fractions in algebra.

A factor in arithmetic is a number that divides exactly into the given number, i.e. there is no remainder.

The definition of a factor in algebra is the same but with 'number' replaced by 'expression'.

Discussion points (page 36)

x is not a factor of the numerator $(2x + 2)$ or the denominator $(3x + 3)$.

The correct answer involves factorising both the numerator and the denominator:

$\dfrac{2(x+1)}{3(x+1)}$

Cancelling $(x + 1)$ gives $\dfrac{2}{3}$.

Neither a nor a^2 is a factor of the numerator and denominator.

The correct answer involves factorising to get

$$\frac{(a-3)(a+2)}{(a-3)(a-5)} = \frac{a+2}{a-5}$$

Discussion point (page 36)

Individual terms have been cancelled rather than factors.

The correct answer is

$$\frac{(2n+3)(2n-3)}{(n+1)} \times \frac{(n+1)(n-1)}{(2n+3)}$$

$= (2n-3)(n-1)$

Exercise 2D (page 37)

1 (i) $\frac{1}{2}$

(ii) $\frac{4}{x+8}$

(iii) $\frac{3}{x-y}$

(iv) $\frac{2x}{3y}$

(v) $\frac{1}{3-p}$

(vi) $\frac{2b^2}{5a^2}$

2 (i) $\frac{x-1}{2}$

(ii) $\frac{x}{x-y}$

(iii) $\frac{1}{a-3}$

(iv) $\frac{3}{2}$

(v) $\frac{3x-1}{3}$

(vi) $\frac{x}{2y}$

3 (i) $\frac{b}{2}$

(ii) x

(iii) $\frac{x(x+y)}{y}$

(iv) $\frac{x}{8(x-1)}$

(v) $2(a+1)$

(vi) $\frac{2(2p+q)}{3(2p-q)}$

4 (i) $\frac{(x-2)}{x(x+2)}$

(ii) $\frac{(2x-1)(x+2)}{(2x+1)(x-1)}$

(iii) $4(p+3)$

(iv) $\frac{3(x-1)(x^2-3)}{(x-3)^2}$

(v) $\frac{3(a+2)}{(a-1)(a-2)}$

(vi) $\frac{t^2-1}{2t}$

5 (i) $\frac{7a}{20}$

(ii) $-\frac{7}{3a}$

(iii) $\frac{(m-3n)}{(m+n)(m-n)}$

(iv) $\frac{5(p+2)}{(p-2)(2p+1)}$

(v) $\frac{5x}{2(x-1)(x+4)}$

(vi) $\frac{7a+8}{6(a-1)(a+4)}$

6 (i) $\frac{(5a+1)}{a(a+1)(a-1)}$

(ii) 2

(iii) $\frac{1}{(p+1)(p-1)}$

(iv) $\frac{2(a^2+b^2)}{(a+b)(a-b)}$

(v) $\frac{10-3x}{x^2-4}$

(vi) $\frac{48-3x}{5(x-2)(x+4)}$

7 (i) $\frac{2(x^2+3x+3)}{(x+1)(x+2)(x+3)}$

(ii) $\frac{5x^2-9x-32}{(x+1)(x-2)(x+3)}$

(iii) $-\frac{1}{x(x+1)^2}$

8 (i) $\frac{7t+3}{(t+1)^2}$

(ii) $\frac{1+3y-3x}{(y+x)(y-x)}$

(iii) $\frac{n^3+6n^2+8n+2}{n(n+1)(n+2)}$

Discussion point (page 38)

If you multiplied both the numerator and the denominator, the multiplier would cancel and the fraction would be unchanged.

Discussion point (page 39)

Not all of the left-hand side has been multiplied by 30.

Exercise 2E (page 39)

1 $x = \frac{5}{6}$

2 $a = \frac{5}{8}$

3 $x = -8$

4 $x = -\frac{2}{3}$

5 $p = \frac{18}{13}$

6 $x = 3$

7 $x = -6$

8 $t = 12$

Exercise 2F (page 41)

1 $a = 4$ $b = -6$

2 $c = 2$ $d = 6$

3 $p = 6$ $q = -40$

4 $a = \frac{5}{2}$ $b = -\frac{33}{4}$

5 $p = 9$ $q = 2$

6 $c = \frac{9}{4}$ $d = \frac{1}{2}$

7 $a = 2$ $b = 8$ $c = -3$

8 $a = 5$ $b = 3$ $c = -35$

9 $p = 3$ $q = -2$ $r = 2$

10 $a = 3$ $b = 24$ $c = -47$

11 $a = 2$ $b = 4$ $c = 8$

12 $p = 23$ $q = 2$ $r = 3$

13 (i) $a = 4$ $b = 4$

(ii) $x = \sqrt{y-4} + 4$

14 (i) $p = 3 \quad q = 1 \quad r = -2$

(ii) $x = \sqrt{\dfrac{y+2}{3}} - 1$

Chapter 3

Discussion point (page 43)

(i) Yes (ii) No

Exercise 3A (page 43)

1 (i) -9 (ii) 0.2
(iii) 15 (iv) -1
(v) -1 (vi) 0

2 (i) 12 (ii) 75
(iii) 3 (iv) -4
(v) 12 (vi) -9

3 (i) 1 (ii) -3
(iii) 0 (iv) 2.25
(v) 0 (vi) 4.2

4 (i) $\frac{8}{3}$ (ii) 2
(iii) $-\frac{4}{7}$

5 (i) $6x - 2$ (ii) $3x + 1$
(iii) $3x^2 - 2$

6 (i) $(x^2 - 1)^2$
(ii) $(x - 1)^4$
(iii) x^4

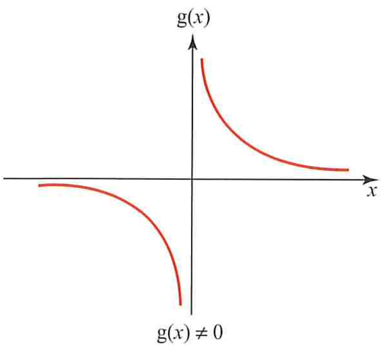
7 (i) $9x^2 + 15x - 1$
(ii) $x^2 + x - 7$

8 (i) 1.5
(ii) 1.2
(iii) 3

Discussion point (page 44)

$\frac{1}{0}$ is not a real number

Activity 3.1 (page 44)



Exercise 3B (page 45)

1 (i) $f(x) < 6$
(ii) $f(x) \geqslant 5$
(iii) $f(x) \geqslant 2$
(iv) $f(x) \geqslant 6$

2 (i) $2 \leqslant f(x) \leqslant 10$
(ii) $-3 < f(x) < 7$
(iii) $f(x) \leqslant 11$
(iv) $-9 < f(x) \leqslant 11$

3 (i) $\frac{5}{2} \leqslant f(x) \leqslant 5$
(ii) $-1.75 \leqslant f(x) \leqslant 0.25$
(iii) $-2.33 \leqslant f(x) \leqslant 3$
(iv) $-7 \leqslant f(x) \leqslant 5$

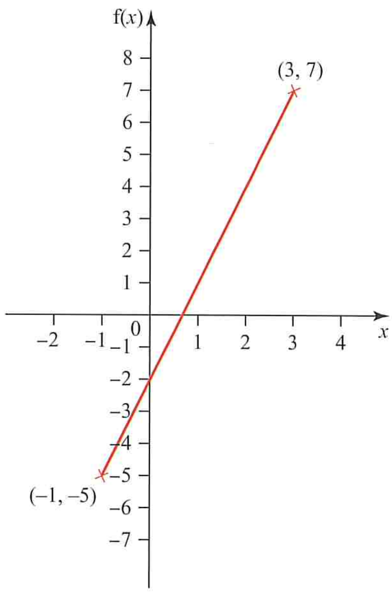
4 (i) $0 \leqslant f(x) \leqslant 4$
(ii) $0 < f(x) < 16$
(iii) $f(x) \geqslant 0$
(iv) $-1 \leqslant f(x) \leqslant 27$

5 (i) $-3 \leqslant f(x) \leqslant 29$
(ii) $-2 \leqslant f(x) \leqslant 46$
(iii) $-5 \leqslant f(x) \leqslant 3$
(iv) $-10 \leqslant f(x) \leqslant 2$

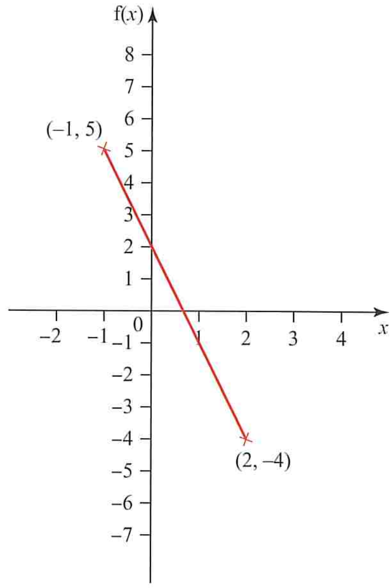
6 (i) $0 \leqslant f(x) \leqslant 15$
(ii) $-1 \leqslant f(x) \leqslant 15$
(iii) $-1 \leqslant f(x) \leqslant 3$
(iv) $0 \leqslant f(x) \leqslant 3$

7 (i) Domain $1 \leqslant x \leqslant 5$
Range $3 \leqslant f(x) \leqslant 8$
(ii) Domain $-4 \leqslant x \leqslant 4$
Range $0 \leqslant f(x) \leqslant 2$
(iii) Domain $-2 \leqslant x \leqslant 3$
Range $0 \leqslant f(x) \leqslant 2$

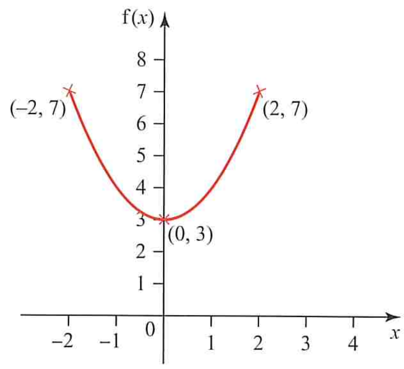
8 (i) $-5 \leqslant f(x) \leqslant 7$



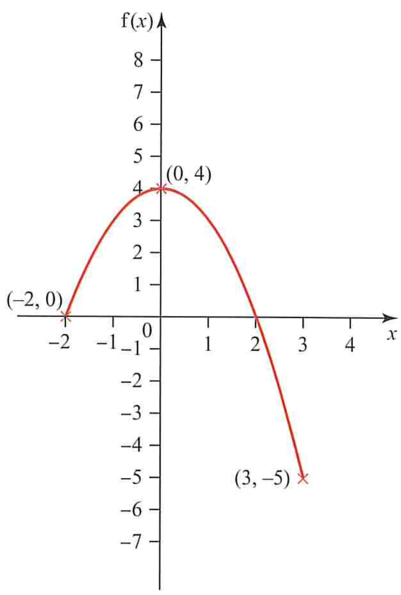
(ii) $-4 \leqslant f(x) \leqslant 5$



(iii) $3 \leqslant f(x) \leqslant 7$



(iv) $-5 \leqslant f(x) \leqslant 4$



Answers

Discussion point (page 47)

$x = -0.5$

Activity 3.2 (page 48)

$f(4) = f(-4) = 16; g(4) = 2;$
$fg(4) = gf(4) = gf(-4) = 4$ but $g(-4)$ and $fg(-4)$ gives Math ERROR

There is no real value for the square root of a negative number.

Exercise 3C (page 48)

1 $f(x) = x^2, g(x) = (3 + x),$
$fg(x) = (3 + x)^2$

2 (i) (a) $fg(x) = 2x^3 - 1$
(b) $gf(x) = (2x - 1)^3$
(ii) (a) $fg(2) = 15$
(b) $gf(2) = 27$
(c) $fg(-3) = -55$
(d) $gf(-3) = -343$

3 (i) (a) $fg(x) = \frac{1}{x^2}$
(b) $fh(x) = (1 - x)^2$
(c) $gf(x) = \frac{1}{x^2}$
(d) $hf(x) = 1 - x^2$
(ii) In this case, $fg(x) = gf(x)$ but $fh(x) \neq hf(x)$

4 $g(x) = x^3$ and $h(x) = 1 - x$

5 (i) $h(x) = x - 2;\ g(x) = \frac{3}{x}$
(ii) $h(x) = x - 2;\ g(x) = \frac{x}{3}$
(iii) $h(x) = 3x - 1;\ g(x) = x^2$
(iv) $h(x) = 3x - 1;\ g(x) = 2^x$

6 (i) $v(x) = 2x$: $u(x) = \sin x$
(ii) $v(x) = \frac{x}{2}$; $u(x) = \cos x$
(iii) $v(x) = x - 30°$; $u(x) = \tan x$
(iv) $v(x) = \sin x;\ u(x) = x^2$

7 (i) $r(x) = x - 2;\ q(x) = x^4;$
$p(x) = 3x$
(ii) $r(x) = 2x;\ q(x) = x + 3;$
$p(x) = \frac{x}{4}$

8 gdfabec

Discussion point (page 51)

Two points on the line *or* one point and the gradient of the line.

Activity 3.3 (page 51)

Line A: 3

Line B: 0

Line C: $-\frac{2}{5}$

Line D: ∞

Discussion point (page 51)

No, since

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}$$

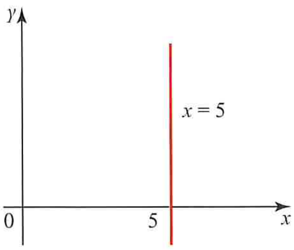
Discussion point (page 54)

(i) $\frac{x}{4} + \frac{y}{3} = 1$
(ii) $a = 4, b = 3$
(iii) a is the intercept on the x-axis and b is the intercept on the y-axis.

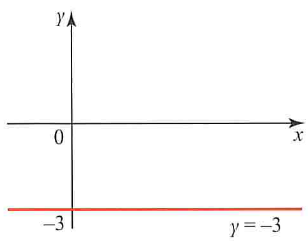
Exercise 3D (page 54)

1 (i) 2 (ii) -3
(iii) $-\frac{11}{5}$ (iv) 3
(v) $7\frac{1}{2}$ (vi) $2\frac{3}{5}$
(vii) $-\frac{1}{5}$ (viii) $-3\frac{2}{3}$

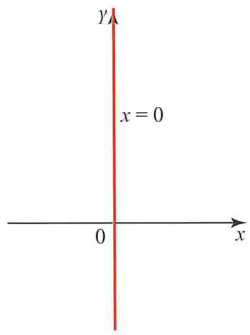
2 (i)



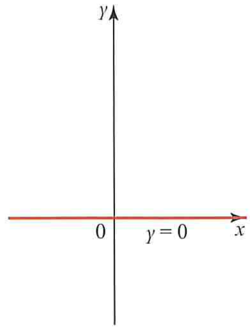
(ii)



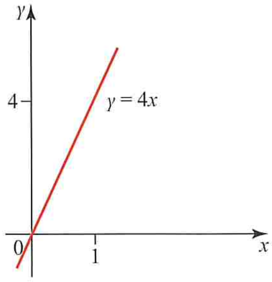
(iii)



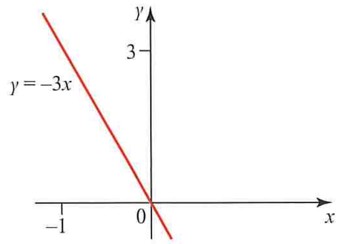
(iv)



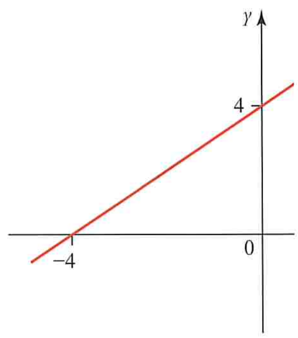
3 (i)



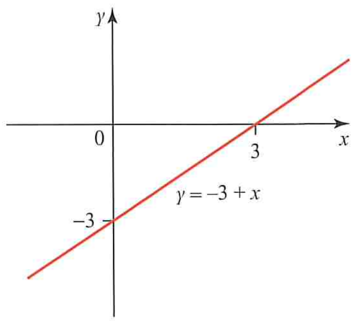
(ii)



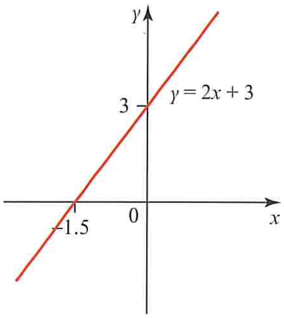
(iii)



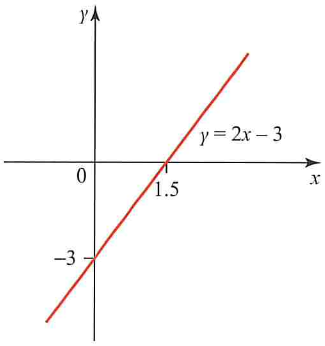
(iv)



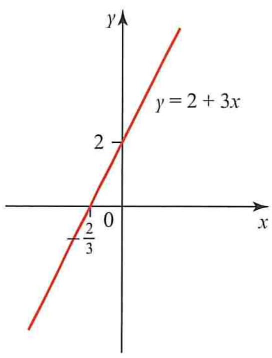
4 (i)



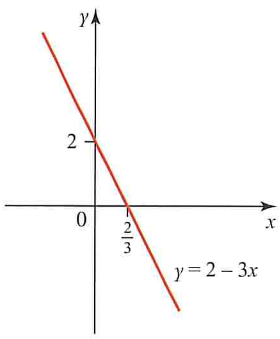
(ii)



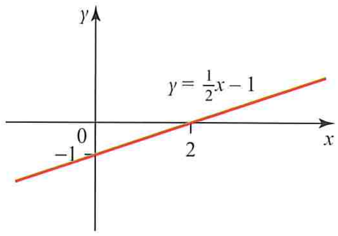
(iii)



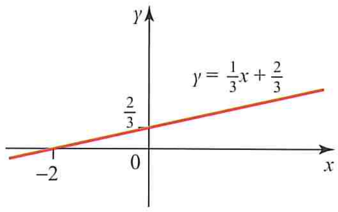
(iv)



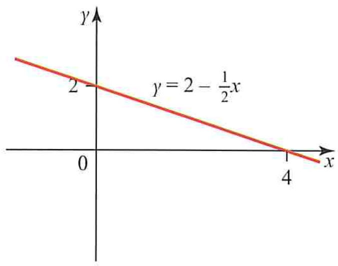
5 (i)



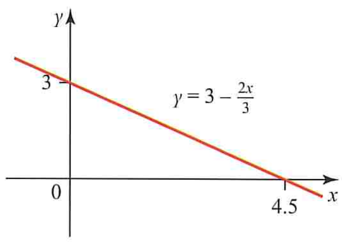
(ii)



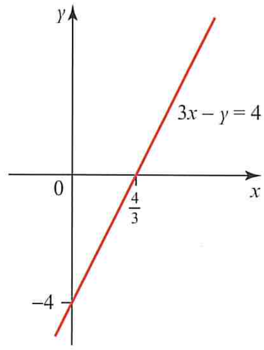
(iii)



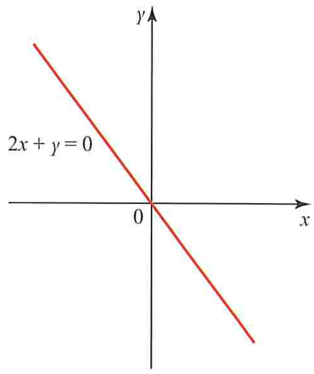
(iv)



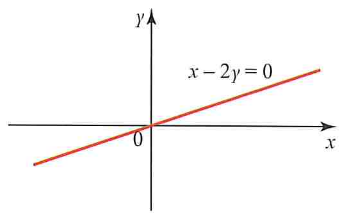
(ii)



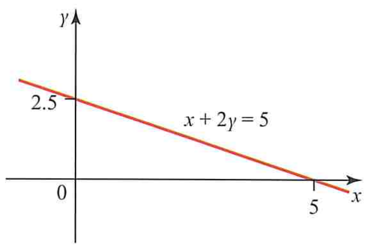
(iii)



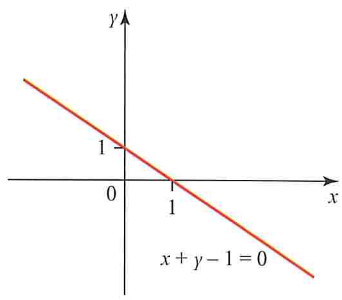
(iv)



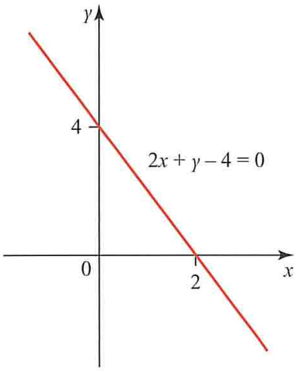
6 (i)



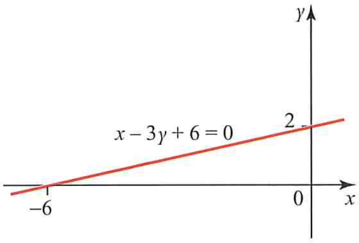
7 (i)



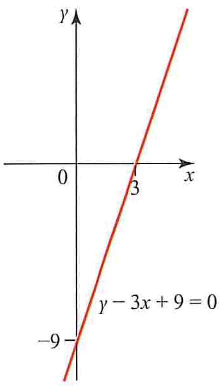
(ii)



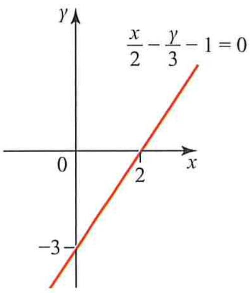
(iii)



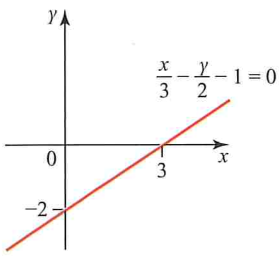
(iv)



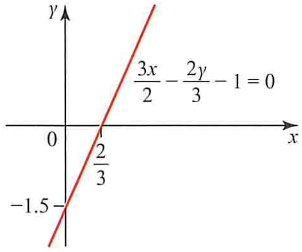
8 (i)



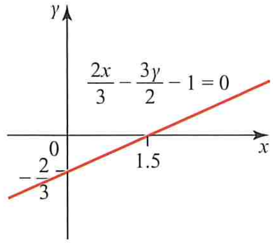
(ii)



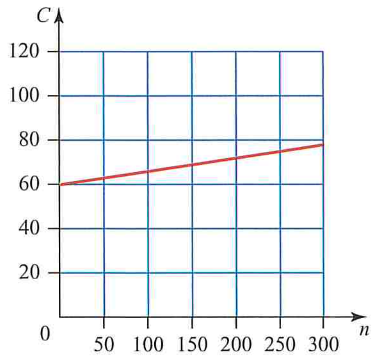
(iii)



(iv)



9 (i) (a) £90 (18p per card)
(b) £360 (7.2p per card)
(ii) £60 set up cost and 6p per card printed.
(iii)



Exercise 3E (page 59)

1 (i) $x = -3$
(ii) $y = 5$
(iii) $y = 2x$
(iv) $2x + y = 4$
(v) $2x + 3y = 12$

2 (i) $x = 5$
(ii) $y = -3$
(iii) $x + 2y = 0$
(iv) $y = x + 4$
(v) $y = 2x - 6$

3 (i) $y = 3x - 7$
(ii) $y = 2x$
(iii) $y = 3x - 13$
(iv) $4x - y - 16 = 0$

4 (i) $y = \frac{1}{3}x$
(ii) $2x - 5y - 42 = 0$
(iii) $3x + 2y + 1 = 0$
(iv) $x + 2y - 12 = 0$

5 (i) $y = x - 2$
(ii) $5x + 3y - 12 = 0$
(iii) $y = x - 5$

6 (i) $3x + 5y - 12 = 0$
(ii) $x + 7y + 32 = 0$
(iii) $y = 2x$

7 (i) $C = 2 + 0.8m$
(ii) £5.20
(iii) 10 miles

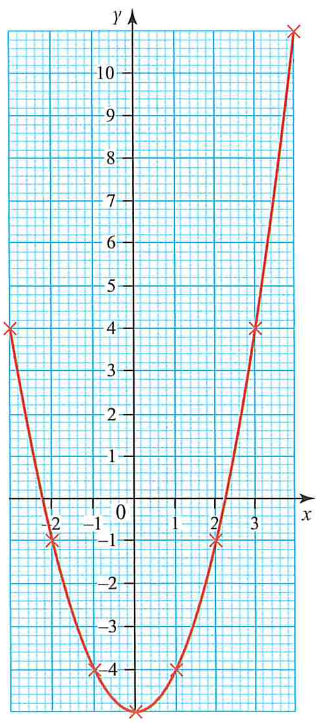
8 (i) $N = 8s + 100$
(ii) £3 030
(ii) Order an extra 80 books instead of 100.

Discussion point (page 60)

A function is of the form $y = f(x)$, enabling you to draw a graph, and the standard form of an equation is $f(x) = 0$, enabling you to find values of x which give the solution of the equation.

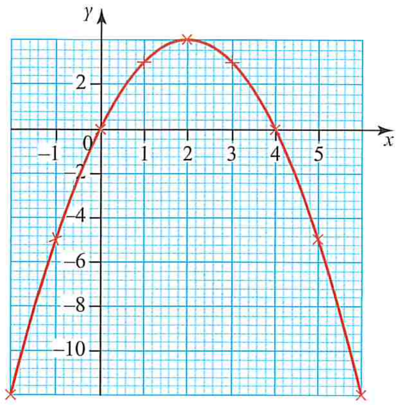
Activity 3.4 (page 60)

x	−3	−2	−1	0	1	2	3	4
y	4	−1	−4	−5	−4	−1	4	11



Activity 3.5 (page 61)

x	-2	-1	0	1	2	3	4	5	6
y	-12	-5	0	3	4	3	0	-5	-12



Exercise 3F (page 63)

1 (i) $y = x^2 - 2x - 3$

(ii) $y = 5 - x^2$

2 (i) $y = 4 - 7x - 2x^2$

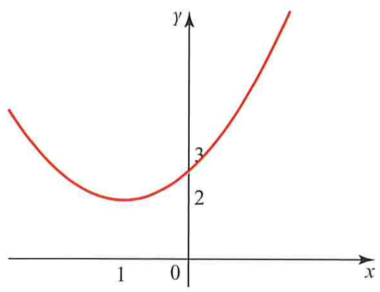
(ii) $y = 4x - x^2$

3 (i) (a) $(-1, 2)$

(b) $x = -1$

(c) $(0, 3)$

(ii)

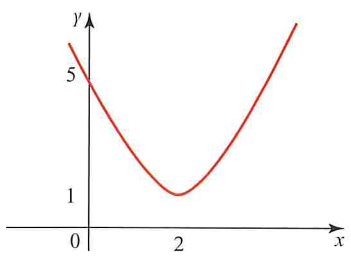


4 (i) (a) $(2, 1)$

(b) $x = 2$

(c) $(0, 5)$

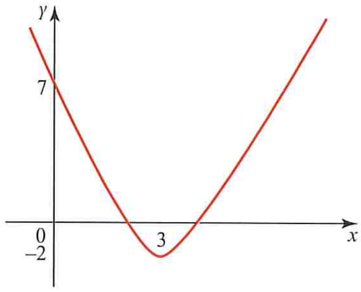
(ii)



5 (i) (a) $(3, -2)$

(b) $x = 3$

(c) $(0, 7)$



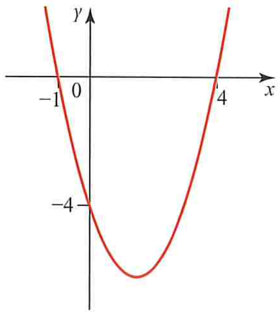
6 (i) (a) $\left(\frac{3}{2}, -\frac{25}{4}\right)$

(b) $x = \frac{3}{2}$

(c) $(0, -4)$

(ii) $(4, 0)$ and $(-1, 0)$

(iii)



7 (i) $x^2 - 6x + 10$

$= (x^2 - 6x + 9) + 1$

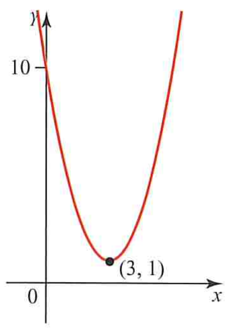
$= (x - 3)^2 + 1$

(ii) $(3, 1)$

(iii) $x = 3$

(iv) $(0, 10)$; it does not intersect the x-axis.

(v)



8 (i) $-(x^2 + 4x - 6)$

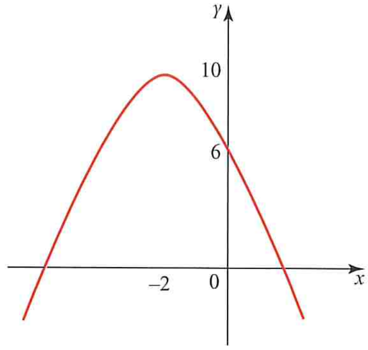
$= -((x + 2)^2 - 10)$

$= 10 - (x + 2)^2$

(ii) $(-2, 10)$

(iii) $x = -2$

(iv) $(0, 6)$, $(-2 - \sqrt{10}, 0)$ and $(-2 + \sqrt{10}, 0)$



9 $y = x^2 - 6x + 2$

10 (i) £19800

(ii) (a) £95 000

(b) £320 000

(c) £500 000

(iii) (a) £19 000

(b) £16 000

(c) £10 000

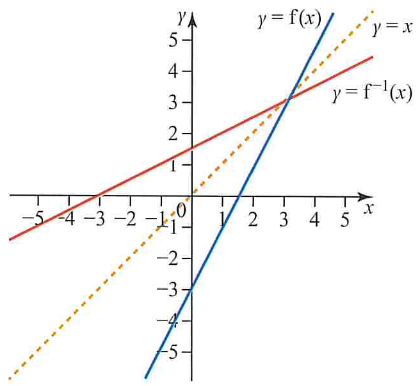
(iv) Would probably make a loss.

Discussion point (page 67)

Exercise 3G extends these basic definitions.

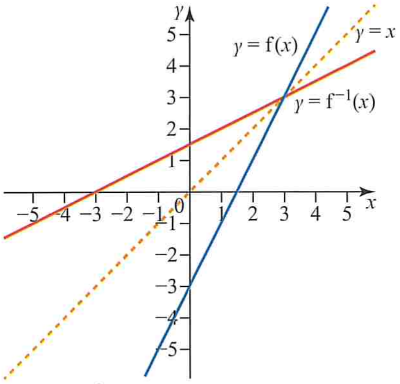
Exercise 3G (page 68)

1 $f^{-1}(x) = \dfrac{x + 3}{2}$

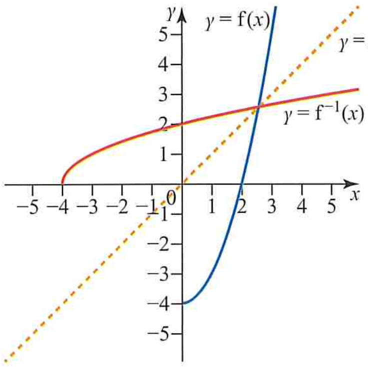


Answers

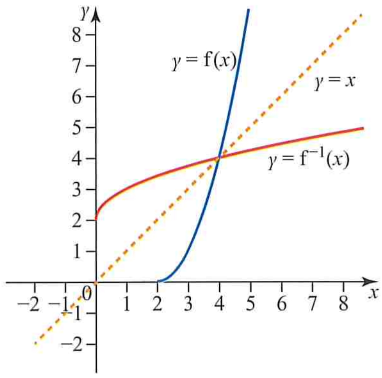
2 $f^{-1}(x) = \dfrac{x+2}{3}$



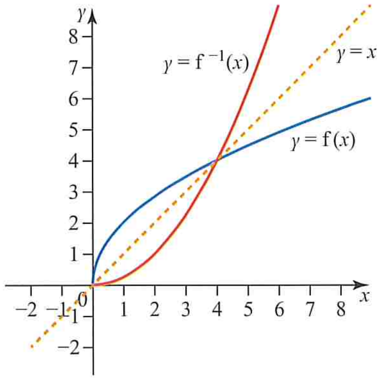
3 $f^{-1}(x) = \sqrt{x+4}$ for $x \geqslant -4$



4 $f^{-1}(x) = \sqrt{x} + 2$ for $x \geqslant 0$

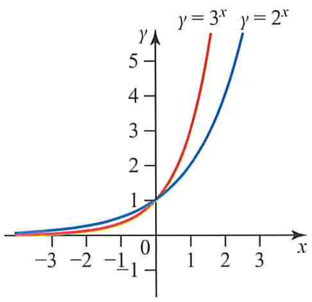


5 $f^{-1}(x) = \dfrac{x^2}{4}$ for $x \geqslant 0$



6 $f^{-1}(x) = \dfrac{1}{x}$ for $x > 0$

NB: Function and inverse are identical.



7 (i) $f^{-1}(x) = \dfrac{x+1}{5}$;

$f^{-1}(4) = 1$

(ii) $f^{-1}(x) = \dfrac{1}{x-2}$ for $x > 2$,

$f^{-1}(3) = 1$

(iii) $f^{-1}(x) = \sqrt{x}$ for $x \geqslant 0$

$f^{-1}(9) = 3$

8 (i)

$f^{-1}(x) = \sqrt{x+3}$ for $x \geqslant -3$

$f^{-1}(-2) = 1$

(ii)

$f^{-1}(x) = \dfrac{\sqrt{x+1}}{2}$ for $x \geqslant -1$

$f^{-1}(3) = 1$

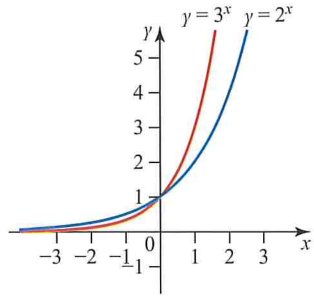
(iii)

$f^{-1}(x) = \dfrac{\sqrt{x-9}}{2}$ for $x \geqslant 9$

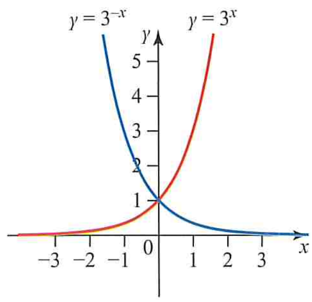
$f^{-1}(13) = 1$

Exercise 3H (page 70)

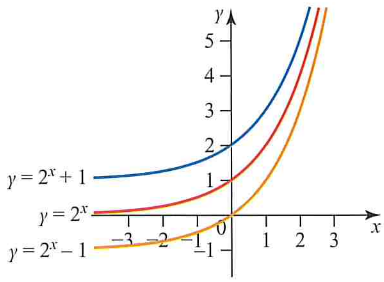
1



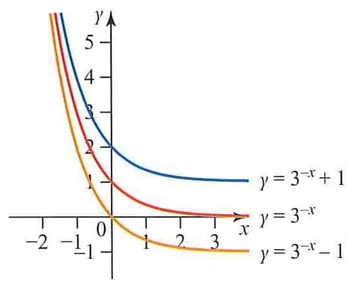
2



3

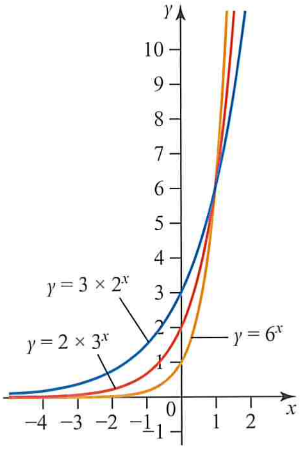


4



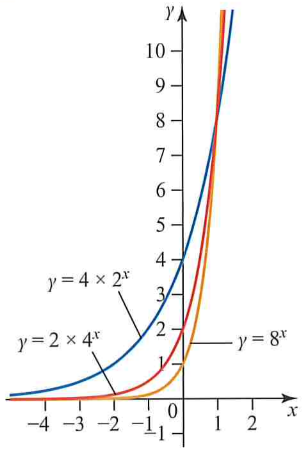
5 (i) 94 (ii) 80 (iii) 5

6



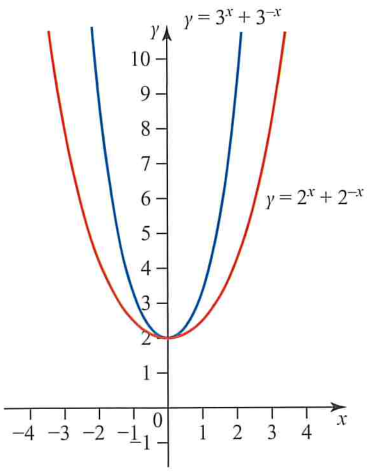
All three graphs intersect at the point (1, 6).

7 (i)



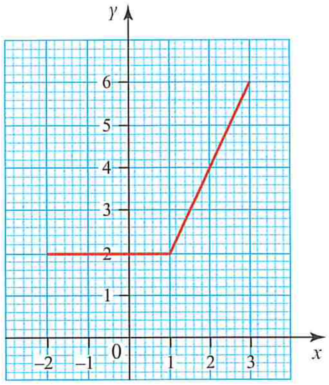
(ii) The shape is similar but the point of intersection is different. It was (1, 6) in Q6 and is (1, 8) in this question.

8 The graphs touch at the point (0, 2).

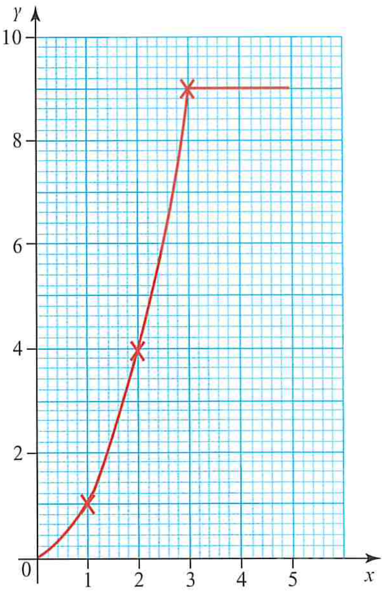


Exercise 3I (page 73)

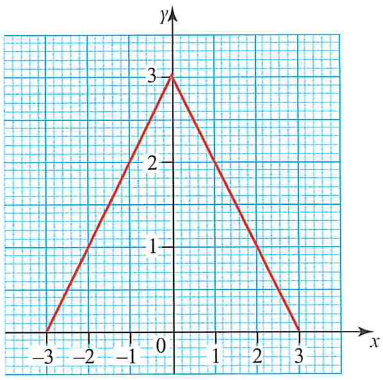
1



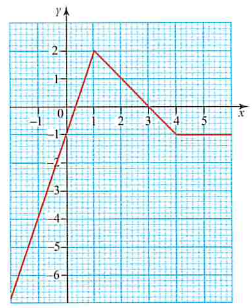
2



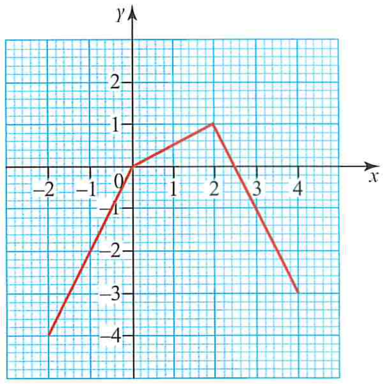
3



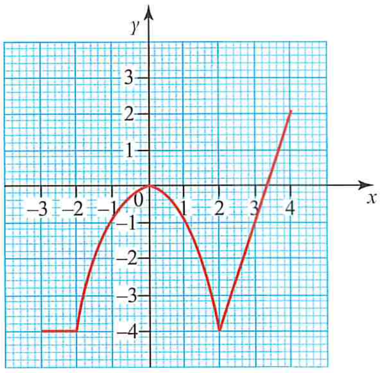
4



5



6



7 (i) $f(x) = x \quad 0 \leqslant x < 3$

$= 3 \quad 3 \leqslant x < 5$

$= 2x - 7 \quad 5 \leqslant x \leqslant 7$

(ii) $0 \leqslant f(x) \leqslant 7$

(iii) $x = 6$

8 (i) $f(x) = 5 \quad -3 \leqslant x < 1$

$= 7 - 2x \quad 1 \leqslant x < 3$

$= x - 2 \quad 3 \leqslant x \leqslant 5$

(ii) $1 \leqslant f(x) \leqslant 5$

(iii) $x = 2, x = 5$

9 (i) $f(x) = x + 3 \quad -3 \leqslant x < 0$

$= 3 \quad 0 \leqslant x < 3$

$= \dfrac{15 - 3x}{2} \quad 3 \leqslant x \leqslant 5$

(ii) $\dfrac{33}{2}$

10 (i) $g(x) = 5x + 15 \quad -3 \leqslant x < -2$
$= 5 \quad -2 \leqslant x < 2$
$= -\frac{5}{2}x + 10 \quad 2 \leqslant x \leqslant 4$

(ii) $\frac{55}{2}$

11 (i) Walking to the bus stop at a steady speed.
(ii) Waiting for the bus.
(iii) Bus is travelling at a constant speed.
Not very realistic – over a distance of 4 km you would expect the bus to have to slow down for traffic lights/make several stops.

Chapter 4

Discussion points (page 79)

If $b^2 - 4ac = 0$ then both answers are the same.

If $b^2 - 4ac < 0$ then no real answers are possible.

However, students who study Further Maths at A-Level will be introduced to a notation which allows us to square root negative numbers. Such numbers are referred to as imaginary. When combined with real numbers, they are referred to as complex numbers, and have many uses in the real world. For example, when modelling the behaviour of electrical circuits, or the flow of air around aeroplane wings.

Exercise 4A (page 82)

1 (i) $x = 2$ or $x = 6$
(ii) $m = 2$ (repeated)
(iii) $p = 5$ or $p = -3$
(iv) $a = -2$ or $a = -9$
(v) $x = -2$ or $x = -\frac{1}{2}$
(vi) $x = 1$ or $x = -1\frac{3}{4}$
(vii) $t = \frac{1}{5}$ or $t = -\frac{1}{3}$
(viii) $r = -\frac{1}{8}$ or $r = -\frac{2}{3}$
(ix) $x = \frac{1}{3}$ or $x = -3$
(x) $p = \frac{2}{3}$ or $p = 4$

2 (i) $x = 4.32$ or $x = -2.32$
(ii) $x = 1.37$ or $x = -4.37$
(iii) $x = 2.37$ or $x = -3.37$
(iv) $x = 1.77$ or $x = -2.27$
(v) $x = 1.68$ or $x = -2.68$
(vi) $x = 2.70$ or $x = -3.70$
(vii) $x = 4.24$ or $x = -0.24$
(viii) $x = 3.22$ or $x = 0.78$

3 (i) $x = -0.23$ or $x = -1.43$
(ii) $x = -0.41$ or $x = -1.84$
(iii) $x = 0.34$ or $x = -5.84$
(iv) $x = 1.64$ or $x = 0.61$
(v) $x = 1.89$ or $x = 0.11$
(vi) $x = -1.23$ or $x = -2.43$

4 3 cm, 4 cm, 5 cm

5 $x = 1.5$

6 9 and 11

7 (i) $t = 1$ s and $t = 2$ s
(ii) 3 seconds

8 (i) $\frac{1}{2}x(2x + 1) = 68$
$x^2 + 0.5x = 68$
$2x^2 + x = 136$
$2x^2 + x - 136 = 0$
(ii) 17 cm

9 (i) (a) $(x + 6)$ cm
(b) $(x - 10)$ cm
(c) $(x - 16)$ cm
(ii) Vol $= 8(x - 16)(x - 10)$
$= 8(x^2 - 10x - 16x + 160)$
$= 8(x^2 - 26x + 160)$
$= 8x^2 - 208x + 1280$
(iii) Length = 34 cm, width = 28 cm

10 (i) $x = \pm\frac{1}{\sqrt{3}}$
(ii) $x = 2,\ x = -\frac{19}{3}$
(iii) $a = -2$
(iv) $p = \frac{13 \pm 3\sqrt{17}}{2}$

11 (i) $p = 3,\ p = \frac{1}{3}$
(ii) $x = 0,\ x = 3$
(iii) $r = 2,\ r = -\frac{7}{3}$

12 (i) $a = 5,\ a = -\frac{19}{2}$
(ii) $x = 7,\ x = 1$
(iii) $x = -2,\ x = -5$

13 30 cm

14 $3x^2 - 4x - 9 = 0$ (or any multiple of this)

Discussion point (page 84)

Infinitely many possibilities.
x can take any value and, in this example, the corresponding value of y is $4 - x$.

Discussion point (page 86)

In this example the correct solution would be found, but in some cases, e.g. if the curve had equation $y^2 = 4x$, additional values that are not part of the solution can be obtained. Always substitute into the equation of the line. For example

$$y = x - 2$$
$$y^2 = 4x - 8$$

has $x = 2, y = 0$ and $x = 6, y = 4$ as its solution.

Substituting into the equation of the curve would also give the incorrect pair of values $x = 6$, $y = -4$, which is not a solution of both equations.

Discussion point (page 87)

Subtract if the coefficients of the variable to be eliminated have the same sign. Add if they have opposite signs.

Exercise 4B (page 88)

1 (i) $x = 5, y = 2$
(ii) $x = 4, y = -1$
(iii) $x = 2\frac{1}{4},\ y = 6\frac{1}{2}$
(iv) $x = -2, y = -3$
(v) $x = 1\frac{1}{2}, y = 4$

(vi) $x = -\frac{1}{2}, y = -6\frac{1}{2}$

2 (i) $x = 2, y = 3$
(ii) $x = 4, y = 3$
(iii) $x = 6, y = 2$
(iv) $x = -\frac{3}{7}, y = 3\frac{2}{7}$
(v) $x = 2, y = 5$
(vi) $x = -1, y = -2$

3 (i) $x = 1, y = 4$ or $x = 4, y = 1$
(ii) $x = 2, y = 3$ or $x = -\frac{2}{3}, y = \frac{1}{3}$
(iii) $x = 4, y = -2$ or $x = -1, y = -7$
(iv) $x = 1, y = 5$ or $x = 11, y = 25$
(v) $x = 4, y = 2$ or $x = -4, y = -2$
(vi) $x = 1, y = -2$ $x = -2\frac{3}{7}, y = -\frac{2}{7}$

4 (i) $3c + 4l = 72, 5c + 2l = 64$; a chew costs 8p and a lollipop costs 12p.
(ii) $x + 5m = 500$, $x + 7m = 660$; $m = 80, x = 100$; £2.60
(iii) $3c + 2n = 145$, $2c + 5n = 225$; $n = 35$, $c = 25$; £1.65
(iv) $2a + c = 3750$, $a + 3c = 3750$; $c = 750$, $a = 1500$; £67.50

5 A(3, 4), B(4, 3)

6 16 and −6

7 (i) (−2, 2)
(ii) Graph (b) because it has one intersection point, whereas graph (a) has no intersection points and graph (c) has two.

Discussion points (page 91)

The answer is zero in both cases.

Discussion points (page 92)

$f(1) = -4$
No, since you would only try factors of the constant term −1.

Exercise 4C (page 95)

1 (i) Factor
(ii) No
(iii) Factor
(iv) Factor
(v) No
(vi) Factor

2 (i) $(x - 1)(x + 1)(x - 3)$
(ii) $(x + 1)(x + 2)(x - 3)$
(iii) $x(x + 1)(x - 2)$
(iv) $(x + 1)(x + 2)(x - 5)$
(v) $(x - 2)(x + 4)(x - 3)$
(vi) $(x + 1)(x - 1)(x - 5)(x + 2)$
(vii) $(x - 1)^4$
(viii) $(x - 2)(x + 2)(x + 3)(x - 3)$
(ix) $x(x + 1)(x - 2)(x + 3)(x - 6)$
(x) $(x - 1)(x + 2)(x - 3)(x + 4)(x - 5)$

3 (i) 1, 3, −2
(ii) 2, −1, −4
(iii) −1, −3, 6
(iv) 1, −1, −4
(v) −2.35, 1, 0.85
(vi) 1, −2, 3, −4, 5

4 −5

5 (i) $1 + p + q + 6 = 0$
(ii) $-27 + 9p - 3q + 6 = 0$
(iii) $p = 0$ $q = -7$

6 (i) $k = -7$
(ii) $x = 1, x = -3$

7 (i) $\frac{8}{x^2}$
(ii) Surface area $= (x \times x) + 4\left(x \times \frac{8}{x^2}\right)$
$= x^2 + 4\left(\frac{8}{x}\right)$
$= x^2 + \frac{32}{x}$
(iii) $x^2 + \frac{32}{x} = 24$
$x^3 + 32 = 24x$
$x^3 - 24x + 32 = 0$
(iv) $x = 4, x = 1.46$

8 $x = 1, x = -\frac{2}{5}, x = \frac{-1 \pm \sqrt{5}}{2}$

Discussion point (page 96)

$8 \times 10^7 \leqslant x \leqslant 3.8 \times 10^8$

Discussion point (page 96)

An equation contains an = sign and any solution will consist of one or more particular values of any variables involved.

An inequality contains any of the signs $<, \leqslant, >, \geqslant$ and any solution will consist of a range of values of its variable(s).

Discussion points (page 96)

An inequality may be rearranged using addition, subtraction, multiplication by a positive number and division by a positive number in the same way as an equation.

Multiplication or division by a negative number reverses the inequality.

$2 < 3$ and $-2 > -3$
$5 > -1$ and $-5 < 1$.

Exercise 4D (page 97)

1 (i) $x < 5$
(ii) $x \geqslant 2$
(iii) $y \leqslant 4$
(iv) $y < 4$
(v) $x \geqslant -3$
(vi) $b \geqslant -3$
(vii) $x > -3$
(viii) $x < 12$
(ix) $x \geqslant -5$
(x) $x \leqslant -4$
(xi) $2 \leqslant x \leqslant 4$
(xii) $2 \leqslant x \leqslant 5$
(xiii) $-3 < x < 1$
(xiv) $1 < x < 2$

2 $-5 \leqslant p - q \leqslant 1$

3 $-1 < x + y < 7$

4 (i) $-2 \leqslant a + b \leqslant 9$
(ii) $-2 \leqslant a - b \leqslant 9$

5 (i) $-4 \leqslant a + b \leqslant 10$
(ii) $-13 \leqslant a - b \leqslant 1$
(iii) $-9 \leqslant 2a + 3b \leqslant 30$

6 (i) always
(ii) never
(iii) sometimes
(iv) sometimes

(v) never
(vi) always

7 (i) always
(ii) always
(iii) never
(iv) sometimes
(v) never
(vi) sometimes

8 $x > 4$

Exercise 4E (page 100)

1 (i) $x < 1$ or $x > 5$
(ii) $-4 \leqslant a \leqslant 1$
(iii) $-1\frac{1}{2} < y < 1$
(iv) $-2 \leqslant y \leqslant 2$
(v) $x < 2$ or $x > 2$
(vi) $1 \leqslant p \leqslant 2$
(vii) $a < -3$ or $a > 2$
(viii) $-4 \leqslant a \leqslant 2$
(ix) $y < -1$ or $y > \frac{1}{3}$
(x) $y \leqslant -1$ or $y \geqslant 5$

2 (i) $-2 < x < 3$
(ii) $-4 < x < 2$
(iii) $-4 < x < -2$
(iv) $2 < x < 3$
(v) $-\frac{1}{2} < x < 1$
(vi) $x < -3, x > 2$
(vii) $x < -3, x > 7$
(viii) $x < -\frac{1}{3}, x > 2$

3 $x > 5$

4 $0 < p < 1$

5 $3\,\text{m} \leqslant$ *length* $\leqslant 7\,\text{m}$

6 $2.5 \leqslant x < 6$

7 $1.5 < t < 2.5$

8 $\frac{1-\sqrt{15}}{2} < x < \frac{1+\sqrt{15}}{2}$

Activity 4.1 (page 101)

(i) $a^3 \times a^0 = a^{3+0} = a^3$; so $a^0 = 1$

(ii) $a^2 \times a^{-2} = a^{2-2} = a^0 = 1$;
so $a^{-2} = \frac{1}{a^2}$

(iii) $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$;
so $a^{\frac{1}{2}}$ is the square root of a.

(iv) $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$
$= a^1 = a$; so $a^{\frac{1}{3}}$ is the cube root of a.

Discussion points (page 104)

Even powers and roots can never be negative. In A-Level Further Maths you will learn how to deal with even powers which are negative.

Odd powers and roots can be negative.

All exponential functions of the form a^x or a^{-x} are always positive.

Exercise 4F (page 105)

1 (i) x^4
(ii) x^{-3}
(iii) $x^{\frac{5}{2}}$
(iv) x^6
(v) x^4
(vi) x^2
(vii) x^{-1}
(viii) x^{10}
(ix) x^2

2 (i) 9
(ii) -8
(iii) $\frac{1}{9}$
(iv) $\frac{1}{4}$
(v) $\frac{1}{64}$
(vi) $\frac{1}{2^{\frac{1}{3}}}$
(vii) 8 or -8
(viii) 9
(ix) $\frac{5}{3}$ or $-\frac{5}{3}$
(x) 9
(xi) $\frac{27}{8}$
(xii) $\frac{1}{10}$ or $-\frac{1}{10}$

3 (i) $x^2 + x$
(ii) $x - 1$
(iii) $x - x^3$
(iv) $x^{-5} + x^{-4}$
(v) $x + x^3$
(vi) $x^3 - 1$

4 (i) $x = 4, x = 1$
(ii) $x = \frac{9}{4}, x = 1$
(iii) $x = 9$
(iv) $x = 1, x = 25$
(v) $x = \frac{1}{4}$
(vi) $x = 1$

5 (i) $x = \pm 2, x = \pm 3$
(ii) $x = \pm 4$
(iii) $x = \pm 2$
(iv) $x = \pm\frac{3}{2}$
(v) $x = \pm 3$

6 (i) $x = -1, x = 2$
(ii) $x = 8, x = -1$
(iii) $x = 1, x = 2$
(iv) $x = 3, x = 2$
(v) $x = 1$
(vi) $x = 3$

7 $x = 2, y = 3$

8 (i) $x = 4$
(ii) $x = -3$
(iii) $x = \frac{5}{2}$
(iv) $x = \frac{3}{2}$
(v) $x = 4$

9 (i) $x = 0, x = -2$
(ii) $x = -2, x = 5$
(iii) $x = \pm 2, x = 4$
(iv) $x = \pm 4, x = 1, x = 2$
(v) $x = \pm 2, x = 4$

Activity 4.2 (page 106)

$(3x^2 - 11x + 7)^{9x^2+27x+18} = 1$ is an example of such an equation, with $x = \pm 1, x = -2, x = 3, x = \frac{2}{3}, x = \frac{8}{3}$

Exercise 4G (page 108)

1 $2(m + 7) - 2(5 + m)$
$= 2m + 14 - 10 - 2m$
$= 4$
$=$ a positive integer

2 $5(c - 3) + 3(c + 7)$
$= 5c - 15 + 3c + 21$
$= 8c + 6$
$= 2(4c + 3)$
$= 2 \times$ an integer
$=$ an even number

3 $(y+6)(y+3)-y^2$
$= y^2+9y+18-y^2$
$= 9y+18$
$= 9(y+2)$
$= 9 \times$ an integer
$=$ a multiple of 9

4 (i) $f(n+1) = (n+1)^2$
$= n^2+2n+1$

(ii) $f(n+1)+f(n-1)$
$= (n+1)^2+(n-1)^2$
$= n^2+2n+1+n^2-2n+$
$= 2n^2+2$
$= 2(n^2+1)$
$= 2 \times$ an integer
$=$ an even number

(iii) $f(n+1)-f(n-1)$
$= (n+1)^2-(n-1)^2$
$= n^2+2n+1-(n^2-2n+1)$
$= 4n$
$= 4 \times$ an integer
$=$ a multiple of 4

5 (i) x^2+2x+5
$= (x+1)^2-1^2+5$
$= (x+1)^2+4$

(ii) $(x+1)^2 \geqslant 0$
$\Rightarrow (x+1)^2+4 \geqslant 4$
$\Rightarrow x^2+2x+5 \geqslant 4$
$\therefore x^2+2x+5>0$

6 $(y-5)^2 \geqslant 0$
$\Rightarrow (y-5)^2+1 \geqslant 1$
$\Rightarrow y^2-10y+26 \geqslant 1$
$\therefore y^2-10y+26>0$

7 $9m^2(3m-1)+(3m)^2$
$= 27m^3-9m^2+9m^2$
$= 27m^3$
$= (3m)^3$
$= (\text{integer})^3$
$=$ a cube number

8 $\dfrac{6p-18}{2p-6} = \dfrac{6(p-3)}{2(p-3)}$
$= \dfrac{6}{2}$
$= 3$
$=$ a positive integer

9 $\dfrac{a^2+ab}{ab+b^2} = \dfrac{a(a+b)}{b(a+b)}$
$= \dfrac{a}{b}$
$= \dfrac{\text{positive}}{\text{negative}}$
$=$ negative

10 $f(4x) = (4x)^2+2\times 4x$
$= 16x^2+8x$
$= 8x(2x+1)$

Exercise 4H (page 110)

1 (i) $4n+6$
(ii) $7n-5$
(iii) $2n-7$
(iv) $25n-25$
(v) $8n-19$
(vi) $0.5n+2.5$
(vii) $50-10n$
(viii) $10-3n$
(ix) $1\frac{1}{2}-\frac{1}{2}n$
(x) $-2.5-1.5n$

2 (i) 589
(ii) −308
(iii) −1792

3 250.5

4 $9-2n+6-3n=15-5n$

5 (i) $p=-52$ $q=18$
(ii) $36n-88$

6 If 44 is a term in the sequence, then $7n-3=44$ for a positive integer value of n
$\Rightarrow 7n = 47$
$\Rightarrow n = \dfrac{47}{7}$
$\Rightarrow n = 6\frac{5}{7}$ which is not an integer
$\therefore$ 44 is not a term in the sequence

7 The difference between consecutive terms is
$(n+1)^3-n^3$
$= n^3+3n^2+3n+1-n^3$
$= 3n^2+3n+1$
$= 3(n^2+n)+1$
$= 3 \times$ an integer $+1$
$=$ one more than a multiple of 3
$\therefore$ the difference is never a multiple of 3

8 nth term
$= n^2-40n+405$
$= (n-20)^2-400+405$
$= (n-20)^2+5 \geqslant 5$
$\therefore$ all terms are positive

Exercise 4I (page 112)

1 (i) n^2+2n+1
(ii) n^2+3n-4
(iii) n^2+6n-3
(iv) $3n^2+4n+1$
(v) $2n^2+3n-1$
(vi) $2n^2-6n$
(vii) $-n^2+2n+10$
(viii) $-2n^2+100$

2 (i) $4n-3$
(ii) $16n^2-24n+9$

3 (i) n^2+2n-1
(ii) n^2+2n+2

4 (i) n^2-4n-2
(ii) $3n^2-12n-6$
(iii) $3n^2-12n+9$

5 (i) 77

(ii) 1

Only the first term will be even. All other terms are the result of adding 3 to an even number which will always produce an odd term.

6 $n^2 - 2n + 12$

7 $n^2 - 2n - 4$

Activity 4.3 (page 113)

Most people would guess that the 6th term is 32.

However, it's actually 31.

The nth term is **not** 2^{n-1} as you might expect.

In fact, the nth term is

$$\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

Activity 4.4 (page 113)

(i)

1.5	2	2.25	2.4
2.5	2.571	2.625	
2.667	2.7	2.727	2.75
2.769	2.786	2.8	2.813

(ii)

2.857	2.903	2.927
2.941	2.970	2.985
2.994		

(iii) The terms are increasing in size and appear to be getting closer to 3.

Exercise 4J (page 114)

1 (i) $\frac{2}{3}$ $\frac{3}{5}$ $\frac{4}{7}$

(ii) 12th

2 (i) 15th

(ii) $\frac{4n-1}{2n-5} = 1$

$4n - 1 = 2n - 5$

$2n = -4$

$n = -2$

n has to be a positive integer

3 (i) 2

(ii) 1

(iii) $\frac{1}{3}$

(iv) $\frac{1}{2}$

(v) $\frac{3}{4}$

(vi) -1

(vii) $-\frac{1}{2}$

(viii) 3

4 $n\text{th term} = \frac{5n+1}{2n+1}$

$$= \frac{5 + \frac{1}{n}}{2 + \frac{1}{n}} \rightarrow \frac{5+0}{2+0}$$

as $n \rightarrow \infty$

$\therefore$ the limit is $\frac{5}{2}$

5 $n\text{th term} = \frac{10-6n}{8n-3}$

$$= \frac{\frac{10}{n} - 6}{8 - \frac{3}{n}} \rightarrow \frac{0-6}{8-0}$$

$= -0.75$ as $n \rightarrow \infty$

$\therefore$ the limit is $\frac{-6}{8} = -0.75$

6 2

7 $\frac{19}{3}$

8 $\frac{2}{7}$

Exercise 4K (page 117)

1 (i) $x = 2, y = 1, z = 5$

(ii) $x = 3, y = -1, z = 2$

(iii) $x = 4, y = 1, z = -3$

2 (i) $x = 2, y = -2, z = 5$

(ii) $x = 4, y = -2, z = 3$

(iii) $x = 7, y = -1, z = -3$

$x = 2, y = -1, z = \frac{1}{2}$

3 (i) $a = -5, b = 4, c = -3$

(ii) $p = -3, q = 4, r = -1$

(iii) $\alpha = 5, \beta = -5, \gamma = 3$

4 (i) $x = 4, y = -5, z = 6$

(ii) $x = 7, y = -1, z = 2$

(iii) $x = 2, y = 1, z = -3$

5 $a = 7, b = 2, c = -3$

6 (i) $a + b + c = 7,$
$4a + 2b + c = 9,$
$9a + 3b + c = 13$

(ii) $n\text{th term} = n^2 - n + 7$

7 $2n^2 - 3n + 7$

8 $(1, 3, 4)$

Activity 4.5 (page 119)

(i) Two planes would never meet if they were parallel.

(ii) Two planes meet at a line, so there are an infinite number of common points.

(iii) e.g. $3x + 2y - z = d$ where d is any constant other than 5.

(iv) No – there could be three different, but parallel, lines on each pair of planes.

(v) Yes – they could all share the same common line, giving an infinite number of common points.

Chapter 5

Discussion point (page 122)

When the increase in x is the same for both lines, then the increase in y is also the same for both lines.

Activity 5.1 (page 122)

(i) As Figure 5.2

(ii) $\angle ABE = \angle BCD$ and
$\angle BCD + \angle CBD = 90°$
$\Rightarrow \angle ABE + \angle CBD = 90°$
i.e. $\angle ABC = 90°$

(iii) $m_1 = \frac{q}{p}$ and $m_2 = -\frac{p}{q}$

(iv) Follows from (iii).

Discussion point (page 124)

$\sqrt{4a^2 + 16b^2}$

$= \sqrt{4(a^2 + 4b^2)}$

$= 2\sqrt{a^2 + 4b^2}$

Exercise 5A (page 125)

1 (i) (a) $-\frac{1}{2}$

(b) $\sqrt{80} = 4\sqrt{5}$

(c) $(6, 7)$

(ii) (a) $\frac{1}{3}$

(b) $\sqrt{90} = 3\sqrt{10}$

(c) $\left(1\frac{1}{2}, 8\frac{1}{2}\right)$

(iii) (a) $\frac{5}{11}$

(b) $\sqrt{146}$

(c) $\left(7\frac{1}{2}, -2\frac{1}{2}\right)$

(iv) (a) $-\frac{1}{3}$

(b) $\sqrt{490} = 7\sqrt{10}$

(c) $\left(-2\frac{1}{2}, -3\frac{1}{2}\right)$

(v) (a) $-\frac{2}{15}$

(b) $\sqrt{229}$

(c) $\left(7, 7\frac{1}{2}\right)$

(vi) (a) $-\frac{5}{13}$

(b) $\sqrt{194}$

(c) $\left(\frac{1}{2}, 2\frac{1}{2}\right)$

(vii) (a) 5

(b) $\sqrt{26}$

(c) $\left(-\frac{1}{2}, -6\frac{1}{2}\right)$

(viii) (a) $\frac{3}{11}$

(b) $\sqrt{130}$

(c) $\left(5\frac{1}{2}, 1\frac{1}{2}\right)$

2 (i) Gradient AB = −1; gradient AC = 1; product = −1

(ii) AB = $\sqrt{32}$; AC = $\sqrt{8}$; BC = $\sqrt{40}$; $BC^2 = AB^2 + AC^2$

3 Gradient AB = $-\frac{1}{2}$; gradient AC = 2; product = −1; AB = AC = $\sqrt{20}$

4 (i) 19.73 units

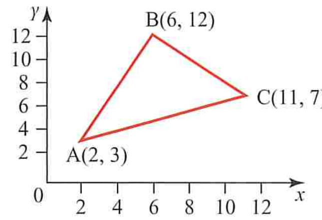
(ii) 9 $units^2$

5 (i) PQ $\sqrt{173}$; QR $\sqrt{173}$; RS $\sqrt{173}$; PS $\sqrt{173}$

(ii) $\left(3\frac{1}{2}, \frac{1}{2}\right)$

(iii) Gradient PQ = $-\frac{2}{13}$; gradient QR = $-\frac{13}{2}$, so PQ is not perpendicular to QR; rhombus

6 (i)



(ii) AB = AC = $\sqrt{97}$; BC = $\sqrt{50}$

(iii) $\left(8\frac{1}{2}, 9\frac{1}{2}\right)$

(iv) 32.5 $units^2$

7 (i) $\left(-\frac{1}{2}, 2\right)$

(ii) (0, −1)

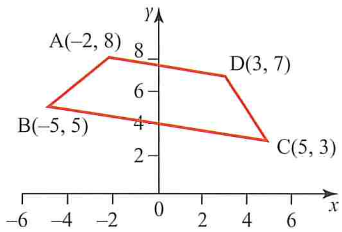
8 (i) Gradient AB = −2; gradient BC = $\frac{1}{2}$

(ii) (7, 4)

9 (i) $q = 2$

(ii) 1 : 2

10 (i)



(ii) Gradient AD = gradient BC = $-\frac{1}{5}$; Gradient AB ≠ gradient DC.

(iii) (8, 6)

Exercise 5B (page 129)

1 (i) 0, ∞; perpendicular

(ii) 2, −2; neither

(iii) $-\frac{1}{2}$, 2; perpendicular

(iv) 1, 1; parallel

(v) −4, −3; neither

(vi) −1, 1; perpendicular

(vii) $\frac{1}{2}, \frac{1}{2}$; parallel

(viii) $-\frac{1}{3}$, 3; perpendicular

(ix) $\frac{1}{2}$, −2 perpendicular

(x) $-\frac{2}{3}, -\frac{2}{3}$; parallel

(xi) $-\frac{1}{3}$, −3; neither

(xii) $\frac{2}{5}, -\frac{5}{2}$; perpendicular

2 (i) $y = 3x - 10$

(ii) $y = 2x + 7$

(iii) $y = 3x - 16$

(iv) $y = 4x - 20$

(v) $3x + 2y - 5 = 0$

(vi) $x + 2y - 10 = 0$

3 (i) $x + 2y = 0$

(ii) $x + 3y - 12 = 0$

(iii) $y = x - 4$

(iv) $x + 2y + 1 = 0$

(v) $2x - 3y - 6 = 0$

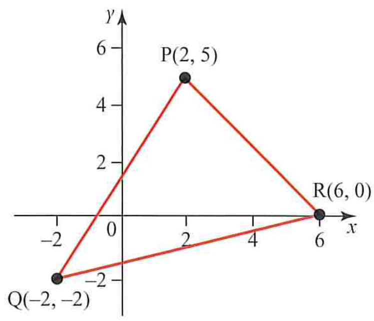
(vi) $x - 2y - 2 = 0$

4 (i) 4

(ii) (4, 3)

(iii) $x + 4y - 16 = 0$

5 (i)



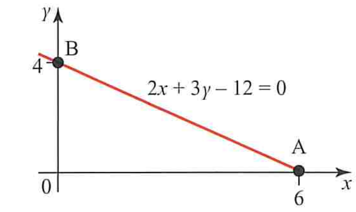
(ii) L$\left(0, 1\frac{1}{2}\right)$, M(2, −1), N$\left(4, 2\frac{1}{2}\right)$

(iii) LR: $x + 4y - 6 = 0$
MP: $x = 2$
NQ: $3x - 4y - 2 = 0$

(iv) Substitute $x = 2$ *and* $y = 1$ into the three equations
LR: $x + 4y - 6$
$= 2 + 4 - 6 = 0$
MP: $x = 2$

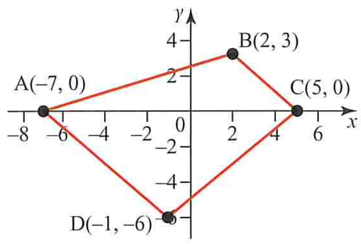
NQ: $3x - 4y - 2$
$= 6 - 4 - 2$
$= 0$

6 (i)



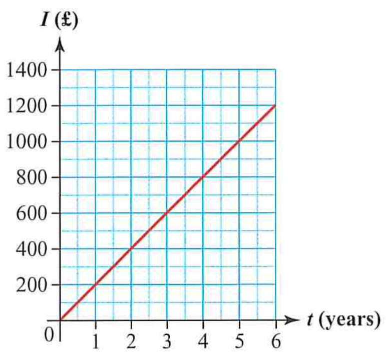
(ii) A(6, 0), B(0, 4)
(iii) 12 units^2
(iv) $3x - 2y = 0$
(v) AB $= \sqrt{52}$ units; shortest distance = 3.33 units (2 d.p.)

7 (i)



(ii) AB: $\frac{1}{3}$; BC: -1; CD: 1; DA: -1
(iii) AB: $x - 3y + 7 = 0$
BC: $x + y - 5 = 0$
CD: $x - y - 5 = 0$
DA: $x + y + 7 = 0$
(iv) AB: $3\sqrt{10}$ units
BC: $3\sqrt{2}$ units
CD: $6\sqrt{2}$ units
DA: $6\sqrt{2}$ units
(v) 54 units^2

8 (i) £200 after each year
(ii) $I = 200t$



(iii) 5 years

9 (i) 125 g
(ii) 7.5 cm

(iii) 80 cm – it is likely that the spring would have reached its elastic limit by then (i.e. it would have been stretched too much and does not function as a spring any more)

Discussion point (page 131)

You need to choose a scale that makes it easy to plot the points and read off the coordinates of the point of intersection. It is particularly difficult to get an accurate solution when it is not represented by a point on the grid.

Discussion points (page 132)

You can always join two points with a straight line. Using three points alerts you if one of your calculated points is wrong.

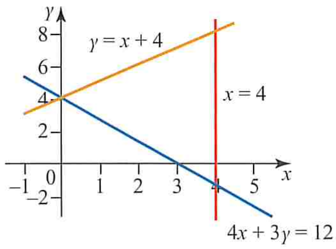
They won't intersect if they are parallel.

Exercise 5C (page 132)

1 (i) $x = 1, y = 0$
(ii) $x = -1, y = 4$

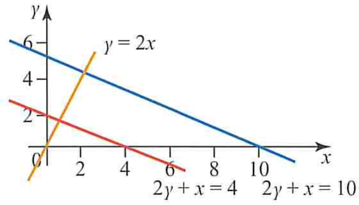
2 (i) $x = 3, y = 2$
(ii) $x = \frac{1}{2}, y = -2$

3 (i)



(ii) (4, 8) at the intersection of $x = 4$ and $y = x + 4$
$\left(4, -1\frac{1}{3}\right)$ at the intersection of $x = 4$ and $4x + 3y = 12$
(0, 4) at the intersection of $y = x + 4$ and $4x + 3y = 12$
(iii) $18\frac{2}{3}$ units^2

4 (i)

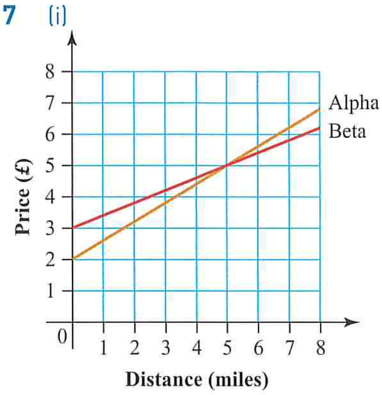


The lines look parallel. They have the same gradient.
(ii) This line looks perpendicular to the first two lines. The first two lines have a gradient of $-\frac{1}{2}$ and the third line has a gradient of +2. The product of the gradients is -1 so they are perpendicular.
(iii) $\left(\frac{4}{5}, 1\frac{3}{5}\right)$ at the intersection of $y = 2x$ and $2y + x = 4$
(2, 4) at the intersection of $y = 2x$ and $2y + x = 10$

5 (i) AB = AC = $3\sqrt{2}$; BC = 6
(ii) AB: $y = x + 3$; AC: $y = 3 - x$; BC: $x = 3$
(iii) Isosceles

6 (i) AB: 3; BC: $-\frac{1}{3}$; CD: 3; DA: $-\frac{1}{3}$. The opposite sides are parallel and adjacent sides are perpendicular
(ii) AB = BC = $\sqrt{10}$
(iii) A square.

7 (i)

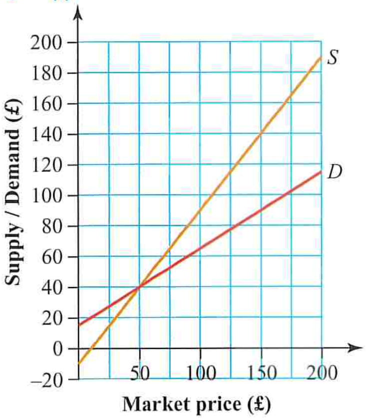


(ii) A: $y = 2 + 0.6x$
B: $y = 3 + 0.4x$

(iii) Beta

(iv) 5 miles

8 (i)



(ii) Equilibrium price = £50; Number bought and sold = 40

Activity 5.2 (page 134)

AE = $x_2 - x_1$ BE = $y_2 - y_1$

$$\frac{AC}{AB} = \frac{p}{p+q}$$

Triangles ACD and ABE are similar.

$$\frac{AD}{AE} = \frac{AC}{AB} \text{ so } \frac{AD}{x_2 - x_1} = \frac{p}{p+q}$$

$$AD = \frac{p}{p+q}(x_2 - x_1)$$

x-coordinate of C is

$$x_1 + \frac{p}{p+q}(x_2 - x_1)$$

$$= \frac{(p+q)x_1 + p(x_2 - x_1)}{p+q}$$

$$= \frac{(px_1 + qx_1 + px_2 - px_1)}{p+q}$$

$$= \frac{qx_1 + px_2}{p+q}$$

Also

$$\frac{CD}{BE} = \frac{AC}{AB} \text{ so } \frac{CD}{y_2 - y_1} = \frac{p}{p+q}$$

$$CD = \frac{p}{p+q}(y_2 - y_1)$$

y-coordinate of C is

$$y_1 + \frac{p}{p+q}(y_2 - y_1)$$

$$= \frac{(p+q)y_1 + p(y_2 - y_1)}{p+q}$$

$$= \frac{(py_1 + qy_1 + py_2 - py_1)}{p+q}$$

$$= \frac{qy_1 + py_2}{p+q}$$

Discussion points (page 135)

1 : 2

1 : 4

Exercise 5D (page 135)

1 (i) (5, 12)

(ii) (9, 1)

(iii) (4, 1)

(iv) $\left(\frac{7}{5}, 17\right)$

(v) (–10, –11)

2 (i) (14, 8)

(ii) (–2, 9)

(iii) (1, 3)

(iv) $\left(-1, -\frac{7}{5}\right)$

(v) (7, 3)

3 (i) 5 : 4

(ii) (7, –8)

4 $\left(-\frac{3}{2}, 2\right)$

5 $\left(\frac{13}{4}, 4\right)$

6 (i) (a) 10 cm by 8 cm

(b) 6 cm by 4 cm.

(ii) 2500 : 1

7 (i) 58 cm

(ii) Height = 52 cm, width = 86 cm

8 (i) $4\sqrt{2}$

(ii) A′B′ = 2, B′C′ = 2, C′A′ = $2\sqrt{2}$

(iii) Ratio is 1 : 2

(iv) Ratio is 1 : 4

Exercise 5E (page 140)

1 (i) $(x-1)^2 + (y-2)^2 = 9$

(ii) $(x-4)^2 + (y+3)^2 = 16$

(iii) $(x-1)^2 + y^2 = 25$

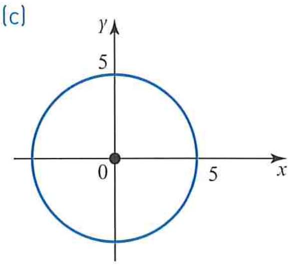
(iv) $(x+2)^2 + (y+2)^2 = 4$

(v) $(x+4)^2 + (y-3)^2 = 1$

2 (i) (a) (0, 0)

(b) 5

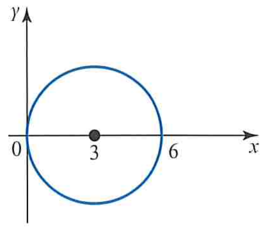
(c)



(ii) (a) (3, 0)

(b) 3

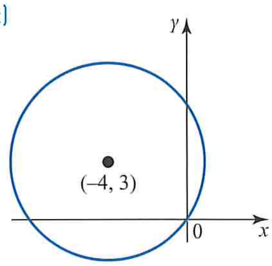
(c)



(iii) (a) (–4, 3)

(b) 5

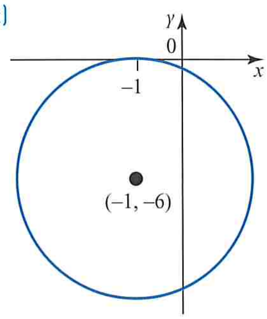
(c)



(iv) (a) (–1, –6)

(b) 6

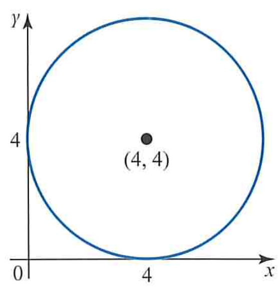
(c)



(v) (a) (4, 4)

(b) 4

(c)



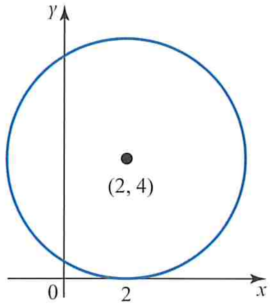
3 $(x-2)^2 + (y+3)^2 = 5$

4 (i) $(3, 1)$

(ii) $\sqrt{26}$

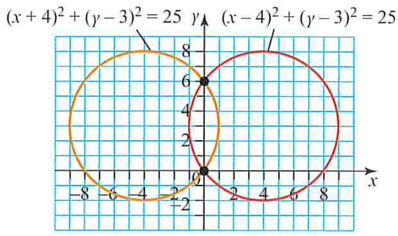
(iii) $(x-3)^2 + (y-1)^2 = 26$

5 Centre $(2, 4)$, radius 4



6 Rewriting the equation gives an apparent radius of $\sqrt{(-14)}$ units

7



Equations are

$(x-4)^2 + (y-3)^2 = 25$ and

$(x+4)^2 + (y-3)^2 = 25$

8 (i) $\sqrt{50}$

(ii) $(x-11)^2 + (y-8)^2 = 50$

(iii) $(16, 13)$

Discussion point (page 141)

In triangles CRM and CSM

CR = CS (radii)

CM is common to both triangles

Angle CMR = angle CMS = 90°

Triangles are congruent (2 sides and a non-included right angle)

Hence RM = MS

Discussion point (page 142)

In triangles TAC and TBC

AC = BC (radii)

TC is common to both triangles

Angle TAC = angle TBC = 90°

Triangles are congruent (2 sides and a non-included right angle)

Hence TA = TB

Exercise 5F (page 144)

1 $\frac{1}{5}$

2 $(5, 0)$

3 $-\frac{3}{2}$

4 $y = -x + 7$

5 $y = \frac{3}{4}x - \frac{25}{4}$

6 $(x-2)^2 + (y-3)^2 = 25$

7 (i) $(x-2)^2 + (y-1)^2 = 25$

(ii) $4y = 3x - 27$

(iii) $x = -3$, $(-3, 1)$

(iv) 10

8 (ii) $x = -5$

(iii) $(-5, -10)$

Chapter 6

Activity 6.1 (page 149)

1 1 4 9 16 25

36 49 64 81 100

121 144 169 196

225 256 289 324

361 400 441 484

529 576 625

$10^2 = 6^2 + 8^2$

$15^2 = 9^2 + 12^2$

$20^2 = 12^2 + 16^2$

$25^2 = 15^2 + 20^2$

$13^2 = 5^2 + 12^2$

$17^2 = 8^2 + 15^2$

$25^2 = 7^2 + 24^2$

Right-angled triangles can be made with sides the lengths of the numbers used.

Exercise 6A (page 152)

1 $x = 28°$ $y = 25°$

2 $x = 113°$

3 $x = 62°$ $y = 48°$

4 $x = 118°$ $y = 18°$

5 $x = 57.5°$

6 $x = 19°$

7 $x = 38°$

8 $c = 42°$

9 $x = 36°$ $y = 18°$

Exercise 6B (page 157)

These solutions may not be unique.

1 Angle ABC = 90 (angle in a semi-circle)

$x + y + 90 = 180$ (angle sum of triangle)

$x = 90 - y$

2 Angle CBF = x (alternate angles)

$x + 2x + y = 180$ (angle sum of triangle)

$3x + y = 180$

3 Angle YAB = b (angles in the same segment)

angle AYB = 90 (angle in a semi-circle)

$a + b + 90 = 180$ (angle sum of triangle)

$a + b = 90$

4 Angle CDB = a (base angles of isosceles triangle)

angle ABF = a (corresponding angles)

angle ABC = a (alternate angles)

angle ABC = angle ABF

5 Angle PTC = 90 (tangent is perpendicular to radius)

angle PCT + 90 + $2y$ = 180 (angle sum of triangle)

angle PCT = $90 - 2y$

angle TMN = $45 - y$ (angle at circumference is half angle at centre)

6 Angle ACB = angle DBA (alternate segment theorem)

angle ACB = angle BAD (base angles of isosceles triangle)

angle DBA = angle BAD

Triangle ABD is isosceles as base angles are equal

7 Angle EDG = $180 - y$ (opposite angles of cyclic quadrilateral)

angle CDG = $130 - y$

angle CGD = $130 - y$ (base angles of isosceles triangle)

$x + 130 - y + 130 - y = 180$
(angle sum of triangle)
$x + 260 - 2y = 180$
$x = 2y - 80$

8 Reflex angle PCR $= 2y$
(reflex angle at centre is double angle at circumference)
angle PCR $= 360 - 2y$
(angles at a point)
Angle CRQ $= x$
(since PQ = QR)
$x + x + y + 360 - 2y = 360$
(angle sum of quadrilateral)
$2x = y$

Discussion point (page 159)

No: since they are defined using the sides of a right-angled triangle they are restricted to
$0 < \theta < 90°$.

Activity 6.2 (page 159)

(i) Around 0.89
(ii) Depends on (i)
(iii) Draw a larger triangle.

Discussion point (page 160)

You need at least 3 decimal places:

$\tan^{-1} 0.714 = 35.5°$, but

$\tan^{-1} 0.71 = 35.4°$.

Discussion point (page 161)

The best function would be tan θ, since this does not use the value of h that you calculated earlier.

Exercise 6C (page 162)

1 (i) 11.2 cm
(ii) 7.7 cm
(iii) 12.1 cm
(iv) 15.1 cm
(v) 6.8 cm
(vi) 7.7 cm

2 (i) 30.6°
(ii) 50.4°
(iii) 55.7°
(iv) 41.4°
(v) 45.0°
(vi) 64.2°

3 (i) 63.6°
(ii) 14.9 cm
(iii) 9.1 cm

4 4.5 m

5 78.2 m

6 282.7 m

7 33.7°

8 (i) 119 km
(ii) 33°
(iii) 333 km h^{-1}

Discussion point (page 164)

The results would be unchanged

Exercise 6D (page 165)

1 (i) $5 + \sqrt{3}$
(ii) $3 + 2\sqrt{3}$
(iii) 5
(iv) $\frac{9}{2}$

2 $\cos 30° = \frac{y}{6\sqrt{3}}$

$\frac{\sqrt{3}}{2} = \frac{y}{6\sqrt{3}}$

$\frac{\sqrt{3}}{2} \times 6\sqrt{3} = y$

$9 = y$

3 $\sin 45° = \frac{\sqrt{8} + \sqrt{2}}{p}$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{8} + \sqrt{2}}{p}$

$p = \sqrt{2}\left(\sqrt{8} + \sqrt{2}\right)$

$p = \sqrt{16} + 2$

$p = 4 + 2$

$p = 6$

4 $(9 + 6\sqrt{3})$ cm^2

5 $(2\sqrt{6})$ cm

6 45°

7 (i) 100 (ii) 1559 (iii) 136

8 (i) $10\sqrt{3}$ m (ii) BC = 10 m; AB = 20 m

Discussion points (page 170)

Undefined means that you cannot find a value for it. When $\theta = 90°$, $x = 0$ and $\cos \theta = 0$, so neither definition works since you cannot divide by zero. $\tan \theta$ is also undefined for $\theta = 90° \pm$ any multiple of 180°.

Discussion point (page 170)

It is a line that is very close to the shape of the curve for large values of x or y.

Discussion points (page 170)

The period is 180° since it repeats itself every 180°.

For $-90° \leqslant \theta \leqslant 0°$, rotate the part of the curve for $0° \leqslant \theta \leqslant 90°$ through 180° about the origin. This gives one complete branch of the curve.

Translating this branch through multiples of 180° to the right or left gives the rest of the curve.

Discussion point (page 171)

The equation will have infinitely many roots since the curve continues to oscillate and the line $y = 0.5$ crosses it infinitely many times.

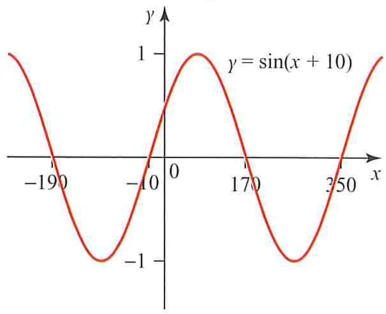
Discussion points (page 171)

$293.6° = -66.4° + 360°$

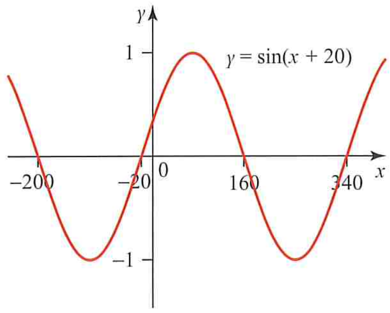
Change the sign of the principal angle (i.e. swap a + for a −, or vice versa) and add 360°.

Alternatively, subtract the principal angle from 360°.

Activity 6.3 (page 173)



- The graph of $y = \sin(x + 10)$ is a translation of the graph of $y = \sin x$ by 10° to the left.



- The graph of $y = \sin(x + 20)$ is a translation of the graph of $y = \sin x$ by 20° to the left.
- The graph of $y = \sin(x + 90)$ is the same as the graph of $y = \cos x$.
- $\cos x \equiv \sin(x + 90)$
  Note: 90 could be replaced with $90 + 360n$, where n is an integer, e.g. 450 or –270
- $\sin x \equiv \cos(x - 90)$
  Note: –90 could be replaced with $-90 + 360n$, where n is an integer, e.g. +270 or –450

Exercise 6E (page 173)

1 (i) 60°, 300°
(ii) 45°, 225°
(iii) 60°, 120°
(iv) 210°, 330°
(v) 90°, 270°
(vi) 101.3°, 281.3°
(vii) 0°, 180°, 360°
(viii) 122.7°, 237.3°
(ix) 90°

2 (i) $\theta = 48.2°$ or $-48.2°$
(ii) $\theta = 45.6°$ or $134.4°$
(iii) $\theta = 69.4°$ or $-110.6°$
(iv) $\theta = -56.4°$ or $-123.6°$
(v) $\theta = 113.6°$ or $-113.6°$
(vi) $\theta = -29.1°$ or $150.9°$

3 (i) $\theta = 60°, 120°, 240°$ or $300°$
(ii) $\theta = 45°, 135°, 225°$ or $315°$
(iii) $\theta = 45°, 135°, 225°$ or $315°$

4 (i) $(2x - 1)(x + 1)$
(ii) $x = 0.5$ or -1
(iii) (a) $\theta = -330°, -210°, -90°, 30°, 150°$ or $270°$
(b) $\theta = -300°, -180°, -60°, 60°, 180°$ or $300°$
(c) $\theta = -333.4°, -225°, -153.4°, -45°, 26.6°, 135°, 206.6°$ or $315°$

5 (i) $x = -180°, -108.4°, 0°, 71.6°$ or $180°$
(ii) $x = -135°, -45°, 45°$ or $135°$
(iii) $x = -180°, -70.5°, 70.5°$ or $180°$
(iv) $x = -150°, -30°$ or $90°$

6 (i) $x = -300°, -120°, 60°$ or $240°$
(ii) $x = -330°, -210°, 30°$ or $150°$
(iii) $x = -315°, -45°, 45°$ or $315°$
(iv) $x = -300°, -240°, 60°$ or $120°$
(v) $x = -315°, -180°, -135°, 0°, 45°, 180°$ or $225°$
(vi) $x = -330°, -30°, 30°$ or $330°$

7 0°, 60°, 300° or 360°

8 (i) 0
(ii) –150°, –141.8°, –38.2° or –30°

Exercise 6F (page 176)

1 (i) (a) $2\sin^2\theta - \sin\theta - 1 = 0$
(b) $\theta = 90°, 210°$ or $330°$
(ii) (a) $\cos^2\theta - \cos\theta - 2 = 0$
(b) $\theta = 180°$
(iii) (a) $2\cos^2\theta + \cos\theta - 1 = 0$
(b) $\theta = 60°, 180°$ or $300°$
(iv) (a) $\sin^2\theta - \sin\theta = 0$
(b) $\theta = 0°, 90°, 180°$ or $360°$
(v) (a) $2\sin^2\theta + \sin\theta - 1 = 0$
(b) $\theta = 30°, 150°$ or $270°$

2 (i) (a) $\cos^2\theta + 2\cos\theta - 2 = 0$
(b) $\theta = 42.9°$
(ii) (a) $\sin^2\theta + \sin\theta - 1 = 0$
(b) $\theta = 38.2°$ or $141.8°$
(iii) (a) $\cos^2\theta + 3\cos\theta - 1 = 0$
(b) $\theta = 72.4°$

3 (i) $\tan\theta = 2$
(ii) $\theta = 63.4°$

4 (i) $\theta = 153.4°$ or $333.4°$
(ii) $\theta = 0°, 30°, 180°, 330°$ or $360°$
(iii) $\theta = 14.5°$ or $165.5°$

5 (i) $\sin^2 x$
(ii) $(1 - \sin^2 x)\sin x$
(iii) $2 - 3\sin x - 2\sin^2 x$

6 $3\sin^2 x + 6\sin x - 6\sin x + 3\cos^2 x$
$= 3\sin^2 x + 3\cos^2 x$
$= 3(\sin^2 x + \cos^2 x)$
$= 3 \times 1$
$= 3$

7 (i) $\tan x\sqrt{1 - \sin^2 x}$
$\equiv \tan x\sqrt{\cos^2 x}$
$\equiv \tan x \cos x$
$\equiv \dfrac{\sin x}{\cos x}\cos x$
$\equiv \sin x$

(ii) $\dfrac{1 - \cos^2 x}{1 - \sin^2 x} \equiv \dfrac{\sin^2 x}{\cos^2 x}$
$\equiv \tan^2 x$

(iii) $(1 + \sin x)(1 - \sin x)$
$\equiv 1 - \sin x + \sin x - \sin^2 x$
$\equiv 1 - \sin^2 x$
$\equiv \cos^2 x$

(iv) $\dfrac{2\sin x\cos x}{\tan x}$
$\equiv \dfrac{2\sin x\cos x}{\frac{\sin x}{\cos x}}$
$\equiv \dfrac{2\sin x\cos^2 x}{\sin x}$
$\equiv 2\cos^2 x$
$\equiv 2(1 - \sin^2 x)$
$\equiv 2 - 2\sin^2 x$

8 $x = 26.6°, 108.4°, 206.6°, 288.4°$

Chapter 7

Exercise 7A (page 183)

1 (i) $9.85\ \text{cm}^2$
(ii) $19.5\ \text{cm}^2$

(iii) $15.2\,cm^2$
(iv) $20.5\,cm^2$

2 $127\,cm^2$

3 (i) $23.8\,cm^2$
(ii) 5.56 cm
(iii) $126\,cm^2$

4 (i) $308\,cm^2$
(ii) 325
(iii) There is likely to be a lot of wastage when tiles are cut for the edges, so he will need more tiles.

5 $173\,cm^2$

6 $\sqrt{8}$ cm or $2\sqrt{2}$ cm or 2.83 cm

7 3.04 cm

8 53.1°

Discussion point (page 184)

It is easier to solve an equation involving fractions if the unknown quantity is in the numerator.

Activity 7.1 (page 186)

$$\frac{\sin z}{6} = \frac{\sin 78°}{8} \Rightarrow z = 47.2°$$

or $z = 132.8°$, but 132.8° is too large to fit into a triangle where one of the other angles is 78°.

Exercise 7B (page 186)

1 (i) 4.61 m
(ii) 11.0 cm
(iii) 5.57 cm
(iv) 7.52 cm

2 (i) 57.7°
(ii) 16.5°
(iii) 103°
(Reject 76.7° since the angle in the diagram is obtuse.)

3 37.9°

4 3.79 km

5 3.59 cm, 3.59 cm, 10.7 cm, 10.7 cm

6 Anna (4.15 km < 4.18 km) – alternatively compare triangle areas

Exercise 7C (page 190)

1 (i) 6.40 cm
(ii) 8.76 cm
(iii) 13.3 cm

2 (i) 41.4°
(ii) 107°
(iii) 90°

3 9.14 cm, 12.3 cm

4 (i) 10 cm
(ii) 112°

5 55.8°

6 13.8 cm

7 $19.0\,cm^2$

8 10.0 km

9 8.54 cm or 4.57 cm

Discussion point (page 192)

If angle A was 118°, then the angles A and B add up to more than 180°.

Alternatively, side BC is shorter than side AC, so angle A must be less than angle B.

Exercise 7D (page 193)

1 12.2 cm

2 6.12 km

3 (i) 26.5 m
(ii) 19.4 m

4 (i) 57.1°, 57.1°, 122.9°, 122.9°
(ii) 14.5 cm

5 (i) 10.2 km
(ii) 117°

6 (i) 29.9 km
(ii) $12.9\,km\,h^{-1}$

7 (i) BD = 2.05 m, EG = 2.07 m
(ii) DE = 4.53 m

8 4.77 km

Discussion point (page 194)

The shortest route is along an arc of a 'great circle', i.e. a circle whose centre is the centre of the earth.

Discussion point (page 196)

One possible example is a ramp used for disabled access to a building.

Discussion point (page 197)

The shelves of a bookcase are parallel; the side of a filing cabinet meets the floor in a line.

Activity 7.2 (page 201)

There is an infinite set of such integers, e.g.

$8^2 + 9^2 + 12^2 = 17^2$ and

$12^2 + 16^2 + 21^2 = 29^2$

Exercise 7E (page 203)

1 (i) 14.1 cm
(ii) 17.3 cm
(iii) 35.3°

2 (i) 3 cm
(ii) 72.1°
(iii) 76.0°

3 (i) 18.4°
(ii) 13 cm
(iii) 17.1°
(iv) Halfway along

4 (i) 75 m
(ii) 67.5 m
(iii) 42°

5 (i) 33.4 m
(ii) 66.7 m
(iii) 115.6 m
(iv) 22.8°

6 (i) 28.3 cm
(ii) 42.4 cm
(iii) 40.6 cm

7 (i) 41.8°
(ii) 219 m
(iii) 186 m
(iv) 51.7°

8 (i) 1.57 m
(ii) 1.547 m
(iii) 3.05 m

9 (i) 15 m
(ii) 16.4 m
(iii) 65.4°
(iv) 69.9°

10 (i) 5.20 cm
(ii) 5.20 cm
(iii) 54.7°
(iv) 16.9 cm

11 (i) 5.20 cm
(ii) $22.0\,cm^2$
(iii) 35.3°

12 (i) 8.49 cm
(ii) 54.7°
(iii) $125\,cm^2$
(iv) $200\sqrt{3}\,cm^2$

13 37.4°

14 10.3 cm
15 29.6°

Chapter 8

Activity 8.1 (page 209)

Taking $R_1 = (2, 4)$, $R_2 = (2.5, 6.25)$, $R_3 = (2.9, 8.41)$, $R_4 = (2.99, 8.9401)$ and $R_5 = (2.999, 8.994\,001)$ gives the gradient sequence 5, 5.5, 5.9, 5.99, 5.999.
Again the sequence seems to converge to 6.

Activity 8.2 (page 210)

(i) The gradient of the chord is $4 + h$. The gradient of the tangent is 4.
(ii) The gradient of the chord is $-2 + h$. The gradient of the tangent is -2.
(iii) The gradient of the chord is $-6 + h$. The gradient of the tangent is -6.
In each case the gradient of the tangent is twice the value of the x-coordinate.

Activity 8.3 (page 211)

Let P be the point (x, x^4) and Q be the point $((x+h), (x+h)^4)$

The gradient of the chord PQ is given by

$$\frac{QR}{PR} = \frac{(x+h)^4 - x^4}{(x+h) - x}$$

$$= \frac{x^4 + 4x^3h + 6x^2h^2 + 4h^3x + h^4 - x^4}{h}$$

$$= \frac{4x^3h + 6x^2h^2 + 4h^2x + h^4}{h}$$

$$= 4x^3 + 6x^2h + 4xh^2 + h^3$$

As Q gets closer to P, $h \to 0$ showing that the gradient of the tangent at (x, x^4) is $4x^3$.

Activity 8.4 (page 212)

In all cases the graphs have a vertical displacement from the origin by an amount equal to the constant term, for all values of x.

Exercise 8A (page 215)

1 (i) $\frac{dy}{dx} = 4x^3$
(ii) $\frac{dy}{dx} = 6x^2$
(iii) $\frac{dy}{dx} = 10x$
(iv) $\frac{dy}{dx} = 63x^8$
(v) $\frac{dy}{dx} = -18x^5$
(vi) $\frac{dy}{dx} = 0$
(vii) $\frac{dy}{dx} = 10$
(viii) $\frac{dy}{dx} = \frac{3}{4}x^2$
(ix) $\frac{dy}{dx} = 2\pi$
(x) $\frac{dy}{dx} = 2\pi x$

2 (i) $\frac{dy}{dx} = 10x^4 + 8x$
(ii) $\frac{dy}{dx} = 12x^3 + 8$
(iii) $\frac{dy}{dx} = 3x^2$
(iv) $\frac{dy}{dx} = 1 - 15x^2$
(v) $\frac{dy}{dx} = 12x^2 + 2$
(vi) $\frac{dy}{dx} = 2$
(vii) $\frac{dy}{dx} = 15x^4$

3 (i) $\frac{dy}{dx} = 15x^4 + 16x^3 - 6x$
(ii) $\frac{dy}{dx} = 5x^4 + 36x^2 + 3$
(iii) $\frac{dy}{dx} = 3x^2 + 84x - 5$

4 (i) $-4x^{-5}$
(ii) $-6x^{-3}$
(iii) $6x - 4x^{-2}$
(iv) $-6x^{-4}$
(v) $2x - 2x^{-3}$
(vi) $-6x^{-3} - 6x^{-4}$

5 (i) $6x - \frac{6}{x^4}$
(ii) $2x - \frac{2}{x^3}$
(iii) $9x^2 - \frac{9}{x^4}$
(iv) $-\frac{2}{x^2} + \frac{6}{x^3}$
(v) $-\frac{1}{2x^2} + \frac{2}{3x^3}$
(vi) $-\frac{2}{3x^2} + \frac{3}{2x^3}$

6 (i) $y = 18x^2$
(ii) $36x$

7 (i) $30t$ cm s^{-1}
(ii) 130 m^2

8 (i) $y = \frac{4}{3}\pi(2x)^3$
$= \frac{4}{3}\pi \times 8x^3$
$= \frac{32}{3}\pi x^3$
(ii) 128π

Exercise 8B (page 216)

1 (i) $3x^2 + 2$
(ii) $18x^2 - 16x$
(iii) $2x + 5$
(iv) $2x + 7$
(v) $12x^2 + 4x^3 - 5x^4$
(vi) $2x - 3$

2 (i) $\frac{5x^4 + 3x^2}{4}$
(ii) $5x^4 + 1$
(iii) $16x^3$
(iv) $6x - 5$
(v) $2x + 1$
(vi) $4x^3$

3 (i) $3x^2 - 2x$
(ii) $6x - 2$

4 54

5 1

6 -7

7 10

8 $\frac{15}{16}$

Exercise 8C (page 219)

1 (i) $\frac{dy}{dx} = 5 - 2x$
(ii) -1
(iii) $x + y - 9 = 0$
(iv) $x - y + 3 = 0$

2 (i) (a) $y = 4$
(b) $x = 2$

(ii) $9x + y = 27$
(iii) $y = 0$
3 (i) $(1, 0)$
(ii) $y = 2x - 2$
(iii) $x + 2y - 1 = 0$
(iv) $Q(0, -2), R(0, \frac{1}{2})$; $1\frac{1}{4}$ units^2
4 (i) $\frac{dy}{dx} = 3x^2 - 6x + 4$
(ii) (a) $y = 4x - 3$
(b) $x + 4y - 22 = 0$
(iii) $x = -1, x = 3$
5 (i) $x + y = 5$
(ii) $x + y = 1$
6 (i) $2p - q = 16$
(ii) $p = 12$
(iii) $(-2, 24)$
(iv) $x - 12y + 96 = 0$
7 (i) $y = 3x - 5$
(ii) $\left(\frac{1}{3}, -1\frac{2}{9}\right)$
8 (i) $2x + y - 15 = 0$
(ii) $x - 2y = 0$
(iii) The normal
9 (i) Substituting $x = 0$
$y = 0 - 0 + 0 = 0$
Substituting $x = 1$
$y = 1.5 - 3.5 + 2 = 0$
(ii) At $(0, 0)$ the tangent is $y = 2x$ and the normal is $x + 2y = 0$.
At $(1, 0)$ the tangent is $x + 2y - 1 = 0$ and the normal is $2x - y - 2 = 0$.
(iii) a rectangle.
10 (i) $\frac{dy}{dx} = 2x - \frac{2}{x^2}$; $(1, 3)$
(ii) Tangent: $y = 3$; normal: $x = 1$
(iii) Tangent: $y = 3.5x - 2$; normal: $2x + 7y = 39$

Discussion point (page 222)

(i) $x + y + 2 = 0$ and $x + y - 2 = 0$
(ii) They are parallel
(iii) $y = x$ and $y = x$
(iv) They are the same line

Exercise 8D (page 222)

1 (i) $x > 0$
(ii) All x values
(iii) $x > -1$
(iv) $x > \frac{3}{2}$
(v) $x > -\frac{2}{3}$
(vi) $x > -2$
(vii) $x < 0$ or $x > \frac{4}{3}$
(viii) $x < -5$ or $x > 1$
(ix) $x < -1$ or $x > 3$
2 (i) $x < 0$
(ii) $x < 3$
(iii) $x < -1$
(iv) $x > 2$
(v) All x values
(vi) $x < -\frac{1}{2}$
(vii) $-2 < x < 0$
(viii) $-3 < x < 4$
(ix) $x < -3$ or $x > 3$
3 $\frac{dy}{dx} = x^2 + 4x + 7 = (x + 2)^2 + 3$
$(x + 2)^2 \geqslant 0$ for all x values.
Adding 3 means always positive, so increasing function.
4 $\frac{dy}{dx} = 3x^2 - 12x + 27$
$= 3(x^2 - 4x + 9)$
$= 3((x - 2)^2 + 5)$
$= 3(x - 2)^2 + 15$
$3(x - 2)^2 \geqslant 0$ for all x values.
Adding 15 means always positive, so increasing function.
5 $x > 1$
6 $\frac{dy}{dx} = -2 - 3x^2$
$-3x^2 \leqslant 0$ for all x values.
Subtracting -2 means always negative, so decreasing function.
7 $\frac{dy}{dx} = -\frac{1}{x^2}$ which is negative for all $x \neq 0$ so the function is decreasing.
8 (i) (a) $x < -1$ and $x > 1$
(b) $-1 < x < 0$ and $0 < x < 1$
(ii) (a) All $x \neq 0$
(b) Never
(iii) (a) $-1 < x < 0$ and $x > 1$
(b) $x < -1$ and $0 < x < 1$
(iv) (a) $x > 0$ (b) $x < 0$
9 (i) $10\,000\text{ cm}^3$
(ii) $1000t\text{ cm}^3$
(iii) $1000\text{ cm}^3\text{s}^{-1}$
(iv) Radius $= 10\left(\sqrt[3]{\frac{(3t)}{(4\pi)}}\right)$

Exercise 8E (page 225)

1 (i) $\frac{dy}{dx} = 9x^2 + 3$
$\frac{d^2y}{dx^2} = 18x$
(ii) $\frac{dy}{dx} = 5x^4$
$\frac{d^2y}{dx^2} = 20x^3$
(iii) $\frac{dy}{dx} = 3 - 20x^3$
$\frac{d^2y}{dx^2} = -60x^2$
2 (i) $\frac{dy}{dx} = 4x^3 - 4x + 5$
$\frac{d^2y}{dx^2} = 12x^2 - 4$
(ii) $\frac{dy}{dx} = 6x^2 + 3$
$\frac{d^2y}{dx^2} = 12x$
(iii) $\frac{dy}{dx} = 3x^2 - 4x$
$\frac{d^2y}{dx^2} = 6x - 4$
3 (i) $\frac{dy}{dx} = 4x + 3$; $\frac{d^2y}{dx^2} = 4$
(ii) $\frac{dy}{dx} = 8x - 4$; $\frac{d^2y}{dx^2} = 8$
(iii) $\frac{dy}{dx} = 11 - 12x$
$\frac{d^2y}{dx^2} = -12$
4 (i) $\frac{dy}{dx} = 9x^2 - 24x + 21$
$\frac{d^2y}{dx^2} = 18x - 24$

(ii) $\frac{dy}{dx} = 8x^3 - 12x^2 + 4x$

$\frac{d^2y}{dx^2} = 24x^2 - 24x + 4$

(iii) $\frac{dy}{dx} = 45x^4 + 24x^3 + 3x^2$

$\frac{d^2y}{dx^2} = 180x^3 + 72x^2 + 6x$

5 (i) $y = 13 - x$

(ii) $P = x(13 - x)$

(iii) $\frac{dy}{dx} = -1$; $\frac{dP}{dx} = 13 - 2x$

(iv) -2

6 (i) $\frac{dy}{dx} = 9x^2 - 4x - 6$

$\frac{d^2y}{dx^2} = 18x - 4$

(ii) 7, −1, 22

(iii) −22, 14, 32

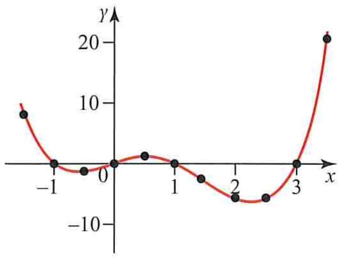
7 (i) $\frac{ds}{dt} = u + at$

$v(12) = u + 12a$

(ii) $\frac{d^2s}{dt^2} = a$

Activity 8.5 (page 226)

(i)



(ii) Decreasing function initially and goes from positive to negative values of y, then gradient is zero before the function increases. Passes through (0, 0) with a positive gradient, has another point with zero gradient before decreasing again and crossing the x-axis to negative values of y. Turns again to go through the x-axis for the 4th time.

Activity 8.6 (page 226)

When $x = 0°$ the gradient is zero. It then decreases through negative values reaching its most negative value when $x = 90°$. It increases to zero when $x = 180°$ and continues to increase through positive values until it is greatest when $x = 270°$. The gradient then decreases to zero when $x = 360°$.

Discussion point (page 228)

There are no more values when $\frac{dy}{dx} = 0$, so there are no more turning points. As x increases beyond the point where $x = 2$, $\frac{dy}{dx}$ takes positive values and so the curve will cross the x-axis again. To the left of $x = -2$ the gradient is always negative, giving a further point of intersection with the x-axis.

Discussion point (page 228)

(i) The curve crosses the x-axis when $x^3 - 12x + 3 = 0$. This does not factorise, so the values of x cannot be found easily.

(ii) Only when the equation obtained when $y = 0$ factorises.

Exercise 8F (page 234)

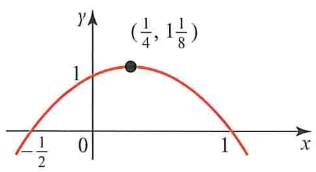
1 (i) (a) $\frac{dy}{dx} = 1 - 4x$; $x = \frac{1}{4}$

(b) $\frac{d^2y}{dx^2} = -4$

(c) Max

(d) $y = 1\frac{1}{8}$

(e)



(ii) (a) $\frac{dy}{dx} = 12 + 6x - 6x^2$;

$x = -1, x = 2$

(b) $\frac{d^2y}{dx^2} = 6 - 12x$.

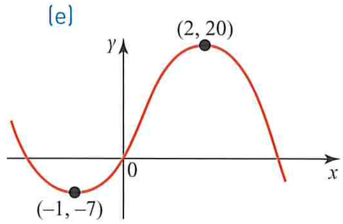
When $x = -1$, $\frac{d^2y}{dx^2} = 18$

When $x = 2$, $\frac{d^2y}{dx^2} = -18$

(c) Min when $x = -1$, max when $x = 2$

(d) $x = -1, y = -7$; $x = 2, y = 20$

(e)



(iii) (a) $\frac{dy}{dx} = 3x^2 - 8x$;

$x = 0, x = 2\frac{2}{3}$

(b) $\frac{d^2y}{dx^2} = 6x - 8$.

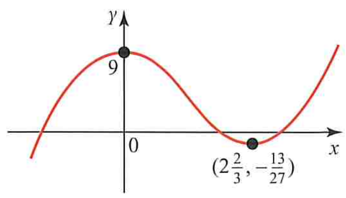
When $x = 0$, $\frac{d^2y}{dx^2} = -8$

When $x = 2\frac{2}{3}$, $\frac{d^2y}{dx^2} = 8$

(c) Max when $x = 0$, min when $x = 2\frac{2}{3}$

(d) $x = 0, y = 9$; $x = 2\frac{2}{3}$, $y = -\frac{13}{27}$

(e)



(iv) (a) $\frac{dy}{dx} = 3x^2 - 4x + 1$;

$x = \frac{1}{3}$; $x = 1$

(b) $\frac{d^2y}{dx^2} = 6x - 4$

When $x = \frac{1}{3}$, $\frac{d^2y}{dx^2} = -2$

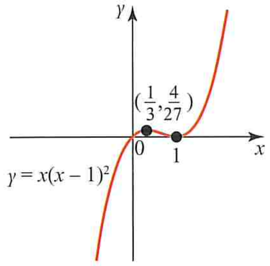
When $x = 1$, $\frac{d^2y}{dx^2} = 2$

(c) Max when $x = \frac{1}{3}$; min when $x = 1$

(d) $x = \frac{1}{3},\ y = \frac{4}{27}$;

$x = 1,\ y = 0$

(e)



(v) (a) $\frac{dy}{dx} = 4x^3 - 6x^2 + 2x$

$x = 0, \frac{1}{2}$ and 1

(b) $\frac{d^2y}{dx^2} = 12x^2 - 12x + 2$

When $x = 0$, $\frac{d^2y}{dx^2} = 2$

When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = -1$

When $x = 1$, $\frac{d^2y}{dx^2} = 2$

(c) Min when $x = 0$

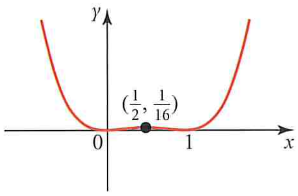
Max when $x = \frac{1}{2}$

Min when $x = 1$

(d) $x = 0,\ y = 0$

$x = \frac{1}{2}, y = \frac{1}{16}$

$x = 1, y = 0$



(vi) (a) $\frac{dy}{dx} = 3x^2 - 48$

$x = -4, x = 4$

(b) $\frac{d^2y}{dx^2} = 6x$

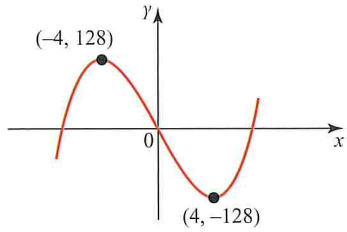
When $x = -4$, $\frac{d^2y}{dx^2} = -24$

When $x = 4$, $\frac{d^2y}{dx^2} = 24$

(c) Max when $x = -4$,
min when $x = 4$

(d) $x = -4, y = 128$;
$x = 4, y = -128$

(e)



(vii) (a) $\frac{dy}{dx} = 3x^2 + 12x - 36$

$x = -6, x = 2$

(b) $\frac{d^2y}{dx^2} = 6x + 12$

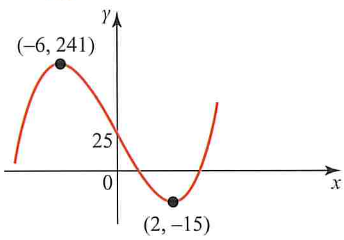
When $x = -6$, $\frac{d^2y}{dx^2} = -24$

When $x = 2$, $\frac{d^2y}{dx^2} = 24$

(c) Max when $x = -6$,
min when $x = 2$

(d) $x = -6, y = 241$
$x = 2, y = -15$

(e)



(viii) (a) $\frac{dy}{dx} = 6x^2 - 30x + 24$

$x = 1, x = 4$

(b) $\frac{d^2y}{dx^2} = 12x - 30$

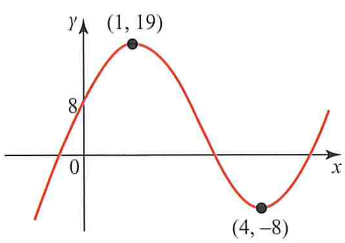
When $x = 1$, $\frac{d^2y}{dx^2} = -18$

When $x = 4$, $\frac{d^2y}{dx^2} = 18$

(c) Max when $x = 1$,
min when $x = 4$

(d) $x = 1, y = 19$
$x = 4, y = -8$

(e)

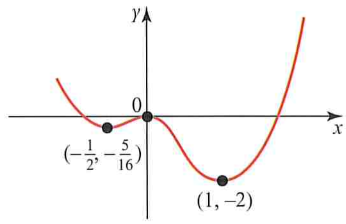


2 (i) $p = 4, q = -3$

(ii) $y = 1\frac{1}{3}, x = \frac{2}{3}$

3 (i) Min at $\left(-\frac{1}{2}, -\frac{5}{16}\right)$, max at $(0, 0)$, min at $(1, -2)$

(ii)



4 (i) $y = \frac{x^2}{9} - \frac{2x}{3} + 2$

5 (i) $q = 12 - p$

(ii) $S = 2p^2 - 24p + 144$

(iii) 72

6 (i) $80a - 2a^2$

(ii) $a = 20,\ b = 20$

(iii) 800

7 (i) $P = x(10 - x)^2$
$= 100x - 20x^2 + x^3$

(ii) $x = 10$ or $x = 3\frac{1}{3}$

(iii) $x = 3\frac{1}{3}$

(iv) $x = 10 \Rightarrow y = 0$ so not a positive number.

8 (i) 2790 metres

(ii) 1220 metres

9 (i) $A = 2x^2 + \frac{864}{x}$ cm²

(ii) $x = 6$ and $y = 6$
$\Rightarrow V = 216$ cm²

$\frac{dA}{dx} = 4x - \frac{864}{x^2}$

$\Rightarrow \frac{d^2A}{dx^2} = 4 + \frac{1728}{x^3}$

> 0 when $x = 6$
so A is a minimum

Chapter 9

Activity 9.1 (page 239)

$\mathbf{AB} = \begin{bmatrix} 14 & -12 \\ 0 & -10 \end{bmatrix}$ and

$\mathbf{BA} = \begin{bmatrix} -4 & 6 \\ 18 & 8 \end{bmatrix}$

Notice that the products **AB** and **BA** are not the same.

Discussion point (page 239)

There is an infinite number of such pairs of matrices.

If either of the matrices (or their product) is a scalar multiple of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then they can be multiplied in either order and the result will be the same, e.g.

$$\mathbf{P} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 4 & -10 \\ -2 & 6 \end{bmatrix}$$

But there are many other cases, e.g.

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \text{ and } \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Exercise 9A (page 240)

1 (i) $\begin{bmatrix} 8 & 12 \\ 4 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 0 \\ -6 & -10 \end{bmatrix}$

(iii) $\begin{bmatrix} 23 \\ 10 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

(v) $\begin{bmatrix} 0 \\ 11 \end{bmatrix}$

(vi) $\begin{bmatrix} 4 \\ -7 \end{bmatrix}$

(vii) $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$

(viii) $\begin{bmatrix} -4 & -6 \\ 7 & 10 \end{bmatrix}$

(ix) $\begin{bmatrix} -12 & 4 \\ 15 & -7 \end{bmatrix}$

(x) $\begin{bmatrix} -18 & -2 \\ 3 & -1 \end{bmatrix}$

(xi) $\begin{bmatrix} 0 & 0 \\ -11 & -14 \end{bmatrix}$

(xii) $\begin{bmatrix} 0 & 0 \\ -3 & -5 \end{bmatrix}$

(xiii) $\begin{bmatrix} 3 & -7 \\ 3 & -3 \end{bmatrix}$

(xiv) $\begin{bmatrix} 0 & 0 \\ -3 & 11 \end{bmatrix}$

2 (i) -1 (ii) 6
(iii) 0.5 (iv) 1.5
(v) 7 (vi) -2

3 (i) $x = 4$ $y = 2$
(ii) $x = -1$ $y = -2$
(iii) $x = 3$ $y = -5$
(iv) $x = -3$ $y = 7$

4 (i) $5x + 3y = 1$ and $2x - y = -4$
(ii) $x = -1$ $y = 2$

5 (i) $a = 3$ $b = 5$
(ii) $a = -2$ $b = 4$
(iii) $a = 1$ $b = -5$
(iv) $a = 2$ $b = 2$

6 $a = 2$ $b = -5$
$c = -1$ $d = 3$

7 $k = 27$

8 $p = 3, q = -2, r = -5$

9 (i) $\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$

(ii) $\mathbf{M} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Discussion point (page 241)

M does not exist if the values of a, b, c and d are such that $ad - bc = 0$

Activity 9.2 (page 242)

(i) (3, 2) (−1, 5) (6, 0) (−3, −4) and (x, y)
No transformation has occurred

(ii) (2, −1) (−4, −3) (0, −4) (−5, 1) and $(x, -y)$
Reflection in the x-axis

Exercise 9B (page 243)

1 (20, 8) **2** (9, −16)
3 (13, −1) **4** −2
5 9
6 $a = -3$ $b = 3$
7 $c = 1$ $d = -2$

8 $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9 5
10 (5, 2)
11 $a = -2, b = 1, c = 8$
12 (i) (2, 4)
(ii) $y = 3x$

Exercise 9C (page 247)

1 (i) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

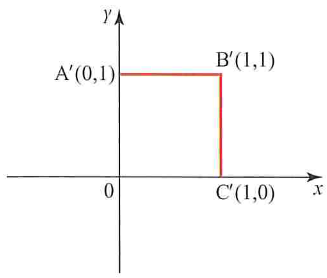
(vi) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(viii) $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

(ix) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

2



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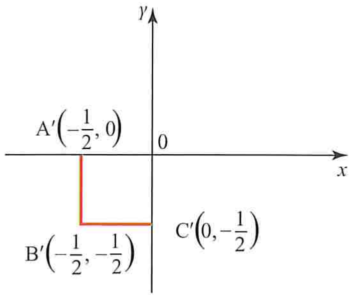
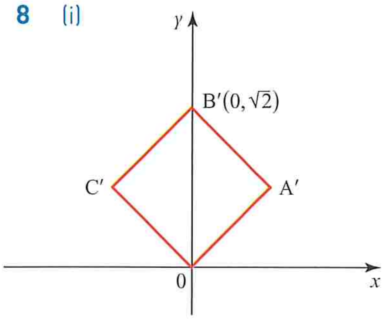


4 (i) Reflection in the y-axis
(ii) Enlargment, scale factor 5, centre (0, 0)
(iii) Reflection in the line $y = x$
(iv) Rotation of 90° anticlockwise about (0, 0)
(v) Rotation of 180° anticlockwise (or clockwise) about (0, 0) OR Enlargement, scale factor −1, centre (0, 0)
(vi) Reflection in the x-axis
(vii) Enlargement, scale factor $1\frac{1}{2}$, centre (0, 0)
(viii) Reflection in the line $y = -x$

5 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

6 16 sq units **7** 8 or −8

8 (i)



(ii) $A' = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$ and $C' = \left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$

(iii) $\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$

Discussion point (page 249)

The following pairs of transformations will produce the same result regardless of the order in which they are applied:

- An enlargement and any other transformation.
- A rotation followed by another rotation.
- A reflection in the x-axis and a reflection in the y-axis.
- A reflection in $y = x$ and a reflection in $y = -x$.

Exercise 9D (page 249)

1 (i) $\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$
(ii) −4

2 (i) $\begin{bmatrix} 3 & -1 \\ 13 & -7 \end{bmatrix}$
(ii) (−7, −41)

3 $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

4 $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

5 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

6 (i) Reflection in the y-axis
(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(iii) Two successive reflections in the y-axis takes you back to the original position.

7 (i) Rotation through 90°, centre (0, 0)
(ii) Rotation through 180°, centre (0, 0)
(iii) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
(iv) Rotation through 270°, centre (0, 0)
(v) A rotation through 180°, centre (0, 0) followed by a rotation through 90°, centre (0, 0) is equivalent to a rotation through 270°, centre (0, 0).
(vi) **ED** is a rotation through 90°, centre (0, 0) followed by a rotation through 180°, centre (0, 0). This is also equivalent to a rotation through 270°, centre (0, 0), the same as **DE**.

8 $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$

$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$ represents an enlargement, centre O, scale factor −6; $k = -6$

9 (i) $\begin{bmatrix} -4 & 5 \\ 5 & -1 \end{bmatrix}$
(ii) (1, −2)

10 Any three 2 × 2 matrices will satisfy the identity **(AB)C** = **A(BC)**.
This is the associative law.
Matrix multiplication is associative.
However, this specification does not assess a candidate's knowledge of the associative law.

Practice questions 1 (page 253)

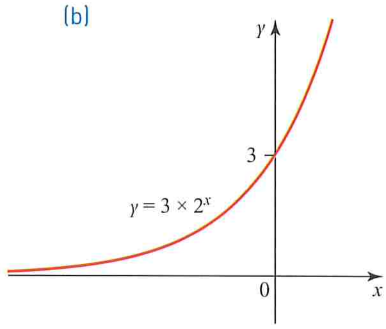
1 $p = 4\frac{1}{2}$

2 $\frac{2}{x + 8}$

3 $\left(\frac{25\pi}{4} - 6\right)$ cm²

4 5000

5 (a) $a = 3, b = 2$
(b)



6 $A\hat{C}B = D\hat{B}C$ (alternate angles)

$C\hat{A}B = D\hat{B}C$ (alternate segmen theorem)

$\therefore A\hat{C}B = C\hat{A}B$

$\therefore$ triangle ABC is isosceles

7 $x^2 + y^2 + 2x - 22y + 22 = 0$ or

$(x+1)^2 + (y-11)^2 = 10^2$

8 (a) 0

(b)

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 - 7\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) - 1$$

$$= \frac{1}{8} + \frac{1}{8} - \frac{7}{4} + \frac{5}{2} - 1$$

$$= 0$$

$\therefore (2x - 1)$ is a factor of $f(x)$

(c) $x = 1,\ x = \frac{1}{2},$

$x = -1 \pm \sqrt{2}$

9 $P = (-2, 4),\ Q = (4, 2)$

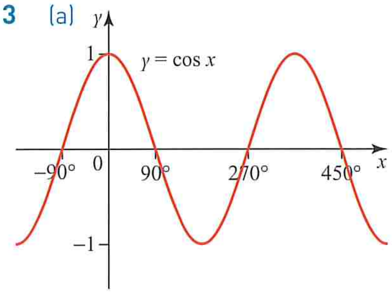
10 $\dfrac{5\sqrt{3} - 7}{2}$

11 (a) $x^{\frac{3}{2}}$

(b) $x = 15\frac{5}{8}$ or $\frac{125}{8}$

12 $p = 12,\quad q = 8$

13 (a)



(b) $x = -30°,\ x = 30°,$

$x = 330°,\ x = 390°$

14 nth term $= \dfrac{3 + \frac{2}{n}}{\frac{1}{n} - 6}$

as $n \to \infty$, then $\frac{1}{n} \to 0$ and $\frac{2}{n} \to 0$

$\therefore$ nth term $\to \dfrac{3+0}{0-6} = -\dfrac{1}{2}$

15 $y = -4x - 24$

16 (a) $a + b + c = -2$

$25a + 5b + c = 2$

$49a + 7b + c = 16$

(b) nth term $= n^2 - 5n + 2$

17 (a) $(y-1)(y-2)$

(b) $y = 1,\quad y = 2$

(c) $x = 1,\quad x = 8$

18 (a) $f^{-1}(x) = \sqrt{x - 6},\ x \geqslant 6$

(b) $x = 1$

19 (a) $n = 7$

(b) $a = 2$

Practice questions 2 (page 256)

1 $y = (x-3)^2$

2 $\dfrac{9y^5}{10x^4}$ or $0.9x^{-4}y^5$

3 $2x^4 - x^3 - 14x^2 + 41x - 28$

4 $3x - 2y = -15$

5 (a) $12x^2 + 2x - 7$

(b) 23

6 18

7 2 : 5

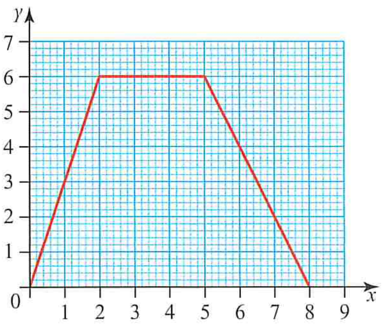
8 (a) AC = 35.5 km (3 s.f.)

(b) 093° (nearest degree)

Note: An answer of 094° is incorrect and probably the result of premature or incorrect rounding.

9 Q = (–4.2, 4.6)

10



11 (a) $A = w(w - y) + xy$

$= w^2 - wy + xy$

$= w^2 + xy - wy$

Note: There are several different methods of proving the required formula.

(b) $y = \dfrac{A - w^2}{x - w}$

12 $x = 54°,\quad y = 18°$

13 (a) 7.84 cm (3 s.f.) or $\dfrac{\sqrt{246}}{2}$ cm

(b) 63.2° (3 s.f.)

(c) 70.3° (3 s.f.)

14 (a) Area

$= \frac{1}{2} \times 3x \times 4x \times \sin 150°$

$= 3x^2$

(b) $0 < x < 6$

15 (a) $\begin{pmatrix} 0 & 1 \\ -6 & -3 \end{pmatrix}$

(b) $P = \left(-1\frac{1}{2},\ 2\right)$

16 $\sin x \tan x \equiv \sin x \times \dfrac{\sin x}{\cos x}$

$\equiv \dfrac{\sin^2 x}{\cos x}$

$\equiv \dfrac{1 - \cos^2 x}{\cos x}$

$\equiv \dfrac{1}{\cos x} - \dfrac{\cos^2 x}{\cos x}$

$\equiv \dfrac{1}{\cos x} - \cos x$

17 (a) $h = \dfrac{1000}{x^2}$

(b) $A = x^2 + 4hx$

$= x^2 + 4\left(\dfrac{1000}{x^2}\right)x$

$= x^2 + \dfrac{4000}{x}$

(c) 476 cm² (3 s.f.)

(d) $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8000}{x^3}$

at $x = 10(2)^{\frac{1}{3}},\ \dfrac{8000}{x^3} > 0$

$\therefore \dfrac{d^2A}{dx^2} > 0$

$\therefore$ A is a minimum.

18 273 cm² (3 s.f.)

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